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Partial ordering of gauge orbit types for SUn-gauge theories

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Abstract

The natural partial ordering of the orbit types of the action of the group of local gauge transformations on the space of connections in space–time dimension $d \leq 4$ is investigated. For that purpose, a description of orbit types in terms of cohomology elements of space–time, derived earlier, is used. It is shown that on the level of these cohomology elements, the partial ordering relation is characterized by a system of algebraic equations. Moreover, operations to generate direct successors and direct predecessors are formulated. The latter allow to successively reconstruct the set of orbit types, starting from the principal type. © 2002 Published by Elsevier Science B.V.

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1. Introduction

The study of geometrical and topological properties of classical non-Abelian gauge theories turned out to be very important for our understanding of non-perturbative aspects of the corresponding quantum field theories. The configuration space of the theory is the gauge orbit space, which is obtained by factorizing the space of connections with respect to the action of the group of local gauge transformations. This space has the structure of a stratified set, because, usually, besides the principal orbit type also non-generic orbit types occur. These may give rise to singularities of the configuration space.

First, the generic, or principal, stratum was investigated—leading to a deeper understanding of the Gribov-ambiguity [14] and of anomalies in terms of index theorems [2].

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In particular, one gets anomalies of purely topological type, which cannot be seen by perturbative quantum field theory [16]. Next, in a paper by Kondracki and Rogulski [11], a systematic study of the structure of the full gauge orbit space was presented. In particular, it was shown that the gauge orbit space is a stratified topological space in the ordinary sense, cf. [10] and references therein.

There are partial results and conjectures concerning the physical relevance of non-generic strata. First of all, non-generic gauge orbits affect the classical motion on the orbit space due to boundary conditions and, in this way, may produce non-trivial contributions to the path integral. They may also lead to localization of certain quantum states, as it was suggested by finite-dimensional examples [6]. Further, the gauge field configurations belonging to non-generic orbits can possess a magnetic charge, i.e. they can be considered as a kind of magnetic monopole configurations, which seem to be related to the quark confinement problem in Chern–Simons theory [1]. Finally, it was suggested in [8] that non-generic strata may lead to additional anomalies.

Most of the problems mentioned here are still awaiting a systematic investigation. In a series of papers, we are going to make a new step in this direction. In [13], we have presented a complete solution to the problem of determining the strata that are present in the gauge orbit space for SUn-gauge theories in compact Euclidean space–time of dimension $d = 2, 3, 4$. The basic idea behind is the 1-1-correspondence between orbit types and equivalence classes of the so-called holonomy-induced Howe subbundles of the principal SUn-bundle, where the gauge connections of the theory under consideration live on. It turns out that Howe subgroups of SUn as well as (holonomy-induced) Howe subbundles can be classified, leading to a classification of orbit types in terms of certain algebraical and topological data. As a first application, we have shown in [13] that—within the context of Chern–Simons theory in $2 + 1$ dimensions—the property of a configuration to be nodal in the sense of Asorey, see [1], is a property of strata. For a given model of this type, the nodal strata can be easily determined.

In [13], one basic problem was left open: the determination of the natural partial ordering in the set of orbit types. In the present paper, we solve this problem. First, in Section 2, we recall the classification of gauge orbit types from Rudolph et al. [13]. In Section 3, we prove that the natural partial ordering is characterized by a system of algebraic equations relating the classifying data via a matrix with non-negative integer entries (inclusion matrix). The inclusion matrix can be visualized by a Bratteli diagram, as explained in Section 4. In Sections 5 and 6, direct successors and direct predecessors are characterized. In particular, operations which generate the direct successors (splitting and merging) and the direct predecessors (inverse splitting and inverse merging) are defined. Finally, an example is discussed: for gauge group SU2 and some space–time manifolds the complete Hasse diagram of the set of orbit types is derived.

2. Classification of gauge orbit types

Let P be a principal SUn-bundle over a compact, connected, orientable Riemannian manifold M of dimension $\dim M \leq 4$. Let $\mathcal{A}^k$ and $\mathcal{G}^k$ denote the sets of connection forms and gauge transformations, respectively, of Sobolev class W^k. Provided $2k > \dim M$, $\mathcal{A}^k$

is an affine Hilbert space and $\mathcal{G}^{k+1}$ is a Hilbert Lie group acting smoothly from the right on $\mathcal{A}^k$ [12,14]. If we view gauge transformations as equivariant maps $P \to \mathrm{SU}n$ then for $A \in \mathcal{A}^k$ and $g \in \mathcal{G}^{k+1}$, the action is given by

$$A^{(g)} = \mathrm{Ad}(g^{-1})A + g^{-1}\,\mathrm{d}g.$$

Let $\mathcal{M}^k$ denote the quotient topological space $\mathcal{A}^k/\mathcal{G}^{k+1}$. This is the gauge orbit space, i.e., the configuration space of our gauge theory. Let $\mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$ denote the set of orbit types of the action of $\mathcal{G}^{k+1}$ on $\mathcal{A}^k$. Recall that orbit types are given by conjugacy classes in $\mathcal{G}^{k+1}$ of stabilizer, or isotropy, subgroups of connections. The set $\mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$ carries a natural partial ordering: let $\tau, \tau' \in \mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$. Then $\tau \leq \tau'$ iff there exist representatives $S, S' \subseteq \mathcal{G}^{k+1}$ of τ, τ', respectively, such that $S \supseteq S'$. Note that this definition is consistent with [3], but not with [11] and several other authors who define the partial ordering inversely. In [11], it was shown that the family $\{\mathcal{M}^k_\tau | \tau \in \mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})\}$, where $\mathcal{M}^k_\tau$ denotes the subset of $\mathcal{M}^k$ of orbits of type τ, is a stratification of $\mathcal{M}^k$ into smooth Hilbert manifolds. For the notion of stratification, see [10] or [11, Section 4.4]. Moreover, for any $\tau \in \mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$, $\mathcal{M}^k_\tau$ is open and dense in the union $\bigcup_{\tau' \leq \tau} \mathcal{M}^k_{\tau'}$. In this sense, the partially ordered set $\mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$ encodes the stratification structure of the gauge orbit space.

In [13], we have derived a description of the elements of $\mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$ in terms of certain cohomology elements of M. In the present paper, we are going to discuss the partial ordering. For the convenience of the reader, we begin with briefly recalling the basic results of Rudolph et al. [13].

A *Howe subgroup* of a group G is a subgroup $H \subseteq G$ that is the centralizer $H = C_G(K)$ of some subset $K \subseteq G$. A *Howe subbundle* of a G-bundle P is a reduction of P to a Howe subgroup. A Howe subbundle is called *holonomy-induced* iff it admits a connected reduction $\tilde{Q}$ to a subgroup $\tilde{H} \subseteq G$, such that

$$\tilde{Q} \cdot C_G(C_G(\tilde{H})) = Q.$$

Let $\mathrm{Howe}_*(P)$ denote the set of isomorphism classes of holonomy-induced Howe subbundles of P factorized by the natural action of the structure group G. Note that here an isomorphism of principal bundles is assumed to commute with the structure group action and to project to the identical mapping on the base space. The set $\mathrm{Howe}_*(P)$ carries a natural partial ordering defined by the relation of inclusion up to isomorphy and up to the action of G.

Proposition 2.1. $\mathrm{Howe}_*(P)$ *is isomorphic, as a partially ordered set, to* $\mathrm{OT}(\mathcal{A}^k, \mathcal{G}^{k+1})$.

Proof. See [13, Theorem 3.3]. □

We note that in the case $G = \mathrm{SU}n$, any Howe subbundle is holonomy-induced, see [13, Theorem 6.2]. Hence, this condition is redundant here.

The following description of $\mathrm{Howe}_*(P)$ has been derived in [13]. First, the Howe subgroups of $\mathrm{SU}n$ were determined. Let $\mathrm{K}(n)$ denote the set of pairs of sequences of strictly positive integers

$$J = (\mathbf{k}, \mathbf{m}) = ((k_1, \dots, k_r), (m_1, \dots, m_r)), \quad r = 1, \dots, n,$$

obeying $\sum_{i=1}^{r} k_i m_i = n$. Let g denote the greatest common divisor of the members of $\mathbf{m}$ and let $\tilde{\mathbf{m}} = (\tilde{m}_1, \dots, \tilde{m}_r)$ be defined by $m_i = g\tilde{m}_i\ \forall i$. We shall always view $\mathbf{k}$ as an $(r \times 1)$-matrix (row vector) and $\mathbf{m}$ as a $(1 \times r)$-matrix (column vector). This turns out to be their natural character. Any $J \in \mathrm{K}(n)$ defines a decomposition

$$\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i},$$

and an embedding

$$\mathrm{M}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{k_r}(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}), \qquad (D_1, \dots, D_r) \mapsto \bigoplus_{i=1}^{r} D_i \otimes \mathbb{1}_{m_i}. \tag{1}$$

Here $\mathrm{M}_l(\mathbb{C})$ stands for the algebra of complex $(l \times l)$-matrices. Identifying $\mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i} \cong \mathbb{C}^{k_i m_i}$, $(c_1, \dots, c_{k_i}) \otimes (d_1, \dots, d_{m_i}) \mapsto (c_1 d_1, \dots, c_{k_i} d_1, \dots, c_1 d_{m_i}, \dots, c_{k_i} d_{m_i})$, the tensor product $D_i \otimes \mathbb{1}_{m_i}$ corresponds to the $(m_i \times m_i)$ block matrix

$$\begin{pmatrix} D_i & 0 & \cdots & 0 \\ 0 & D_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_i \end{pmatrix}.$$

We denote the image of the embedding (1) by $\mathrm{M}_J(\mathbb{C})$ and its intersections with $\mathrm{U}n$ and $\mathrm{SU}n$ by $\mathrm{U}J$ and $\mathrm{SU}J$, respectively. Note that $\mathrm{U}J$ is the image of the restriction of (1) to $\mathrm{U}k_1 \times \cdots \times \mathrm{U}k_r$. By construction, $\mathrm{M}_J(\mathbb{C})$ is a unital $*$-subalgebra of $\mathrm{M}_n(\mathbb{C})$.

Proposition 2.2. *Up to conjugacy, the Howe subgroups of* $\mathrm{SU}n$ *are given by* $\mathrm{SU}J$, $J \in \mathrm{K}(n)$.

Proof. See [13, Lemma 4.1]. □

In order to classify principal $\mathrm{SU}J$-bundles over M, the homotopy classes of maps from M to the classifying space $\mathrm{BSU}J$ have to be determined. Through building the Postnikov tower of $\mathrm{BSU}J$ up to the 5th stage the following was shown.

Proposition 2.3. *Let M be a manifold,* $\dim M \leq 4$ *and let Q, Q' be principal* $\mathrm{SU}J$*-bundles over M. Assume that for any characteristic class α defined by an element of* $H^1(\mathrm{BSU}J, \mathbb{Z}_g)$, $H^2(\mathrm{BSU}J, \mathbb{Z})$, *or* $H^4(\mathrm{BSU}J, \mathbb{Z})$ *there holds $\alpha(Q) = \alpha(Q')$. Then Q and Q' are isomorphic.*

Proof. See [13, Corollary 5.5]. □

A generating set for the characteristic classes mentioned in the proposition can be constructed as follows. Consider the natural homomorphisms

$$j_J : \mathrm{SU}J \to \mathrm{U}J \quad \text{(embedding)},$$

$$\mathrm{pr}^{\mathrm{M}}_{J,i} : \mathrm{M}_J(\mathbb{C}) \to \mathrm{M}_{k_i}(\mathbb{C}) \quad \text{(projection onto the } i\text{th factor)},$$

$$\mathrm{pr}^{\mathrm{U}}_{J,i} : \mathrm{U}J \to \mathrm{U}k_i \quad \text{(projection onto the } i\text{th factor)}.$$

For any positive integer l, let $\gamma_{\mathrm{U}l} = 1+\gamma_{\mathrm{U}l}^{(2)}+\cdots+\gamma_{\mathrm{U}l}^{(2l)}$ denote the sum of generators of the cohomology algebra $H^*(\mathrm{BU}l,\mathbb{Z})$. We assume that the generators are chosen in such a way that for the canonical blockwise embedding $j_l : \mathrm{U}l \to \mathrm{U}(l+1)$ there holds $(\mathrm{B}j_l)^*\gamma_{\mathrm{U}(l+1)} = \gamma_{\mathrm{U}l}\ \forall l$. (Recall that $\mathrm{B}j_l : \mathrm{BU}l \to \mathrm{BU}(l+1)$ is the map between classifying spaces associated to j_l.) Then, in particular, the characteristic classes defined by the generators $\gamma_{\mathrm{U}l}^{(2k)}$ are the kth Chern classes. The cohomology elements

$$(\mathrm{B}j_J)^*(\mathrm{Bpr}_{J,i}^{\mathrm{U}})^*\gamma_{\mathrm{U}k_i},\quad i=1,\ldots,r,$$

of $H^*(\mathrm{BSU}J,\mathbb{Z})$ define characteristic classes

$$\alpha_{J,i} : \mathrm{Bun}(M,\mathrm{SU}J) \to H^*(M,\mathbb{Z}),\qquad Q \mapsto (f_Q)^*((\mathrm{B}j_J)^*(\mathrm{Bpr}_{J,i}^{\mathrm{U}})^*\gamma_{\mathrm{U}k_i}),\tag{2}$$

where $i=1,\ldots,r$. Here $f_Q : M \to \mathrm{BSU}J$ is the classifying map of Q and $\mathrm{Bun}(M,\mathrm{SU}J)$ stands for the set of isomorphism classes of principal $\mathrm{SU}J$-bundles over M. We denote $\alpha_J(Q) = (\alpha_{J,1}(Q),\ldots,\alpha_{J,r}(Q))$.

Next, for any positive integer l, let $j_l : \mathbb{Z}_l \to \mathrm{U}1$ denote the canonical embedding and let p_l denote the endomorphism $z \mapsto z^l$ of U1. We define a homomorphism

$$\lambda_J^{\mathrm{U}} : \mathrm{U}J \to \mathrm{U}1,\qquad D \mapsto \prod_{i=1}^{r} p_{\tilde{m}_i} \circ \det{}_{\mathrm{U}k_i} \circ \mathrm{pr}_{J,i}^{\mathrm{U}}(D).\tag{3}$$

One can check that the diagram

$$\begin{array}{ccc} \mathrm{U}J & \xrightarrow{\ j_J\ } & \mathrm{U}n \\ {\scriptstyle \lambda_J^{\mathrm{U}}}\downarrow & & \downarrow{\scriptstyle \det_{\mathrm{U}n}} \\ \mathrm{U}1 & \xrightarrow[\ p_g\ ]{} & \mathrm{U}1 \end{array}\tag{4}$$

commutes. Moreover, we notice that the image of $\mathrm{SU}J$ under λ_J^{U} is the subgroup $j_g(\mathbb{Z}_g)$ of U1. Thus, λ_J^{U} induces a homomorphism $\lambda_J^{\mathrm{S}} : \mathrm{SU}J \to \mathbb{Z}_g$ by requiring the diagram

$$\begin{array}{ccc} \mathrm{SU}J & \xrightarrow{\ j_J\ } & \mathrm{U}J \\ {\scriptstyle \lambda_J^{\mathrm{S}}}\downarrow & & \downarrow{\scriptstyle \lambda_J^{\mathrm{U}}} \\ \mathbb{Z}_g & \xrightarrow[\ j_g\ ]{} & \mathrm{U}1 \end{array}\tag{5}$$

to commute. (In fact, one can show that λ_J^{S} projects to an isomorphism of the group of connected components of $\mathrm{SU}J$ onto $\mathbb{Z}_g$.)

One can show that the Bockstein homomorphism $\beta_g : H^1(\mathrm{B}\mathbb{Z}_g,\mathbb{Z}_g) \to H^2(\mathrm{B}\mathbb{Z}_g,\mathbb{Z})$, induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_g \to 0$, is an isomorphism, see the proof of Lemma 5.9 in [13]. Thus, we can consider the cohomology element

$$(\mathrm{B}\lambda_J^{\mathrm{S}})^*\beta_g^{-1}(\mathrm{B}j_g)^*\gamma_{\mathrm{U}1}^{(2)}$$

of $H^1(\mathrm{BSU}J,\mathbb{Z}_g)$. It defines a characteristic class

$$\xi_J : \mathrm{Bun}(M,\mathrm{SU}J) \to H^*(M,\mathbb{Z}_g),\qquad Q \mapsto (f_Q)^*((\mathrm{B}\lambda_J^{\mathrm{S}})^*\beta_g^{-1}(\mathrm{B}j_g)^*\gamma_{\mathrm{U}1}^{(2)}).\tag{6}$$

By construction, the characteristic classes α_J and ξ_J are subject to a relation. To formulate it, let us introduce the following notation. Let r', r be positive integers. For any $\Delta \in \mathrm{M}_{r',r}(\mathbb{N})$ (set of $(r' \times r)$-matrices with non-negative integer entries), we define a map

$$E_\Delta : \prod_{i=1}^{r} H_0^{\mathrm{even}}(\cdot, \mathbb{Z}) \to \prod_{i'=1}^{r'} H_0^{\mathrm{even}}(\cdot, \mathbb{Z}),$$

$$(\alpha_1, \ldots, \alpha_r) \mapsto (\alpha_1^{\Delta_{11}} \smile \cdots \smile \alpha_r^{\Delta_{1r}}, \ldots, \alpha_1^{\Delta_{r'1}} \smile \cdots \smile \alpha_r^{\Delta_{r'r}}). \tag{7}$$

Here powers are taken w.r.t. the cup product and $H_0^{\mathrm{even}}(\cdot, \mathbb{Z})$ denotes the subset of $H^{\mathrm{even}}(\cdot, \mathbb{Z})$ of elements of the form $\alpha = 1 + \alpha^{(2)} + \alpha^{(4)} + \cdots$. Note that H_0^{even} is a semigroup w.r.t. the cup product. Let $E_{\Delta,i'}^{(2j)}(\alpha)$ denote the component of degree $2j$ of the i'th member of $E_\Delta(\alpha)$.

Proposition 2.4. *The characteristic classes α_J, ξ_J are subject to the relation*

$$E_{\tilde{\mathbf{m}}}^{(2)}(\alpha_J(Q)) = \beta_g(\xi_J(Q)) \quad \forall Q \in \mathrm{Bun}(M, \mathrm{SU}J). \tag{8}$$

Recall that $\tilde{\mathbf{m}}$ is viewed as a $(1 \times r)$-matrix.

Proof. See [13, Theorem 5.13]. □

We introduce the notation

$$H^{(J)}(\cdot, \mathbb{Z}) = \prod_{i=1}^{r} \{\alpha_i \in H_0^{\mathrm{even}}(\cdot, \mathbb{Z}) | \alpha_i^{(2j)} = 0 \ \text{ for } \ j > k_i\}. \tag{9}$$

Consider the following two equations in the variables $\alpha \in H^{(J)}(M, \mathbb{Z})$, $\xi \in H^1(M, \mathbb{Z}_g)$:

$$E_{\tilde{\mathbf{m}}}^{(2)}(\alpha) = \beta_g(\xi), \tag{10}$$

$$E_{\mathbf{m}}(\alpha) = c(P). \tag{11}$$

Here $c(P)$ denotes the total Chern class of P.

Proposition 2.5. *If* $\dim M \leq 4$, *the characteristic classes α_J and ξ_J define a bijection from* $\mathrm{Bun}(M, \mathrm{SU}J)$ *onto the set of solutions of Eq.* (10). *By restriction, they define a bijection from the subset of* $\mathrm{Bun}(M, \mathrm{SU}J)$ *of reductions of P onto the set of solutions of Eqs.* (10) *and* (11).

Proof. See [13, Theorems 5.14 and 5.17]. □

Note that the content of Eq. (11) in degree 2 is a consequence of Eq. (10).

Let $\mathrm{K}(P)$ denote the disjoint union of the solution sets of Eqs. (10) and (11) over all $J \in \mathrm{K}(n)$. We write the elements of $\mathrm{K}(P)$ as triples $(J; \alpha, \xi)$, where $J \in \mathrm{K}(n)$ and (α, ξ) is a solution of the corresponding equations. According to Proposition 2.5, the set $\mathrm{K}(P)$ classifies the Howe subbundles of P up to isomorphy.

Finally, the action of the structure group SUn on Howe subbundles of P was factored out by passing to the set $\hat{\mathrm{K}}(P)$ that is obtained from $\mathrm{K}(P)$ by identifying $(J;\alpha,\xi)$ with $(\sigma J;\sigma\alpha,\xi)$ for all permutations σ of $1,\dots,r$. Here σJ stands for

$$\sigma J = (\sigma\mathbf{k}, \sigma\mathbf{m}). \tag{12}$$

Theorem 2.6. *The collection of characteristic classes* $\{\alpha_J, \xi_J | J \in \mathrm{K}(n)\}$, *defines, by passing to quotients, a bijection from* $\mathrm{Howe}_*(P)$ *onto* $\hat{\mathrm{K}}(P)$.

Proof. See [13, Theorem 7.2]. □

In the sequel, it is convenient to work with the inverse of this bijection. To construct it, for any $L \in \mathrm{K}(P)$, $L = (J;\alpha,\xi)$, let Q_L denote the isomorphism class of SUJ-subbundles of P defined by

$$\alpha_J(Q_L) = \alpha, \tag{13}$$

$$\xi_J(Q_L) = \xi. \tag{14}$$

Then the pre-image of the element of $\hat{\mathrm{K}}(P)$ represented by L is given by the conjugacy class of Q_L under SUn-action. The (isomorphy classes of) subbundles Q_L may be viewed as some kind of standard representatives of the elements of $\mathrm{Howe}_*(P)$.

To conclude this section, for later use, let us collect some formulae involving the function E_Δ. For any i', one has

$$E^{(2)}_{\Delta,i'}(\alpha) = \sum_{i=1}^{r} \Delta_{i'i}\alpha_i^{(2)}, \tag{15}$$

$$\begin{aligned} E^{(4)}_{\Delta,i'}(\alpha) = {} & \sum_{i=1}^{r} \Delta_{i'i}\alpha_i^{(4)} + \sum_{i=1}^{r} \frac{\Delta_{i'i}(\Delta_{i'i}-1)}{2}\alpha_i^{(2)} \smile \alpha_i^{(2)} \\ & + \sum_{1\le i<j\le r} \Delta_{i'i}\Delta_{i'j}\alpha_i^{(2)} \smile \alpha_j^{(2)}, \end{aligned} \tag{16}$$

see [13, Lemma 5.11]. In particular, for any non-negative integer l,

$$E^{(2)}_{l\Delta,i'}(\alpha) = lE^{(2)}_{\Delta,i'}(\alpha) \quad \forall i'. \tag{17}$$

Taking into account that the cup product is commutative in even degree, one can also check that for any $\Delta \in \mathrm{M}_{r',r}(\mathbb{N})$ and $\Delta' \in \mathrm{M}_{r'',r'}(\mathbb{N})$ there holds

$$E_{\Delta'\Delta} = E_{\Delta'} \circ E_\Delta. \tag{18}$$

3. Characterization of the partial ordering

In this section, we are going to determine the natural partial ordering of $\mathrm{Howe}_*(P)$ on the level of the classifying set $\hat{\mathrm{K}}(P)$.

Let $L=(J;\alpha,\xi)$, $L'=(J';\alpha',\xi')$ be elements of $\mathrm{K}(P)$. Let $[Q_L]$ and $[Q'_L]$ denote the conjugacy classes of Q_L and Q'_L, respectively, under the action of $\mathrm{SU}n$. The natural partial ordering on the set $\mathrm{Howe}_*(P)$ is defined as follows:

$$[Q_L] \le [Q_{L'}] \Leftrightarrow \exists D \in \mathrm{SU}n \quad \text{such that} \quad Q_L \cdot D \subseteq Q_{L'}. \tag{19}$$

Here inclusion is understood up to isomorphy. We aim to express the relation (19) in terms of L and L'.

Let $D \in \mathrm{SU}n$, such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. Then there also holds $D^{-1}\mathrm{U}JD \subseteq \mathrm{U}J'$ and $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$. We have an associated homomorphism

$$h_D^{\mathrm{M}} : \mathrm{M}_J(\mathbb{C}) \to \mathrm{M}_{J'}(\mathbb{C}), \qquad C \mapsto D^{-1}CD,$$

and, derived from that, homomorphisms $h_D^{\mathrm{U}} : \mathrm{U}J \to \mathrm{U}J'$ and $h_D^{\mathrm{S}} : \mathrm{SU}J \to \mathrm{SU}J'$. Due to $\mathrm{M}_J(\mathbb{C})$ and $\mathrm{M}_{J'}(\mathbb{C})$ being finite-dimensional unital C^*-algebras, the embedding h_D^{M} is characterized by an $(r' \times r)$-matrix $\Delta(D) \in \mathrm{M}_{r',r}(\mathbb{N})$ (non-negative integer entries), called *inclusion matrix*. The matrix $\Delta(D)$ can be constructed as follows: for $1 \le i \le r$ and $1 \le i' \le r'$, consider the homomorphism

$$\mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_J(\mathbb{C}) \overset{h_D^{\mathrm{M}}}{\to} \mathrm{M}_{J'}(\mathbb{C}) \overset{\mathrm{pr}^{\mathrm{M}}_{J',i'}}{\to} \mathrm{M}_{k'_{i'}}(\mathbb{C}), \tag{20}$$

where the first map is canonical embedding to the ith factor of $\mathrm{M}_J(\mathbb{C})$. Define $\Delta(D)_{i'i}$ to be the number of fundamental irreps contained in the representation of $\mathrm{M}_{k_i}(\mathbb{C})$ defined by (20).

Lemma 3.1. *Let $J, J' \in \mathrm{K}(n)$. Let $D \in \mathrm{SU}n$ such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. Then*

$$\Delta(D)\mathbf{k} = \mathbf{k}', \tag{21}$$

$$\mathbf{m} = \mathbf{m}'\Delta(D). \tag{22}$$

Conversely, let $\Delta \in \mathrm{M}_{r',r}(\mathbb{N})$ be a solution of (21) *and* (22). *Then there exists $D \in \mathrm{SU}n$ such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$ and $\Delta(D) = \Delta$.*

Proof. First, let D be given as proposed. Consider the representations

$$\mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_J(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}), \tag{23}$$

$$\mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_J(\mathbb{C}) \overset{h_D^{\mathrm{M}}}{\to} \mathrm{M}_{J'}(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}). \tag{24}$$

The numbers of fundamental irreps contained in (23) and (24) are m_i and $\sum_{i'=1}^{r'} m'_{i'}\Delta(D)_{i'i}$, respectively. Since (23) and (24) are isomorphic—a bijective intertwiner being given by D—we obtain (22). Moreover, inserting this equation into $\mathbf{m}\cdot\mathbf{k} = \mathbf{m}'\cdot\mathbf{k}'$ yields $\mathbf{m}'\cdot(\mathbf{k}'-\Delta\mathbf{k}) = 0$. By construction, the members of the sequence $\mathbf{k}'-\Delta\mathbf{k}$ are non-negative. Since the members of $\mathbf{m}$ are strictly positive, Eq. (21) follows.

Conversely, let Δ be a solution of (21) and (22). Consider the decompositions

$$\mathbb{C}^n = \oplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i}, \tag{25}$$

$$\mathbb{C}^n = \oplus_{i'=1}^{r'} \mathbb{C}^{k'_{i'}} \otimes \mathbb{C}^{m'_{i'}} \tag{26}$$

defined by J and J', respectively. Due to (22) and (21), (25) and (26) admit subdecompositions

$$\mathbb{C}^n = \oplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \left(\oplus_{i'=1}^{r'} \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}} \right), \tag{27}$$

$$\mathbb{C}^n = \oplus_{i'=1}^{r'} \left(\oplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \right) \otimes \mathbb{C}^{m'_{i'}}, \tag{28}$$

respectively. There exists $D \in \mathrm{SU}n$ transforming (28) into (27) by a suitable permutation of the subspaces $\mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}}$. One can check that $D^{-1}\mathrm{M}_J(\mathbb{C})D$ leaves the decomposition (26) invariant. It follows $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$, hence $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. Moreover, from (27) and (28), one can read off that $\Delta(D) = \Delta$. □

We remark that for general inclusions of $\mathrm{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{k_r}(\mathbb{C}) \subseteq \mathrm{M}_{k'_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{k'_{r'}}(\mathbb{C})$, inclusion matrices only have to obey $\sum_{i=1}^{r} \Delta_{i'i} k_i \leq k'_{i'}$, where the inclusion is unital iff there holds equality for all i'.

Let us denote the set of solutions of the system of equations (21) and (22) by $\mathrm{N}(J, J')$. We note that if $\mathrm{N}(J, J') \neq \emptyset$, then (22) implies that g' divides g. Hence, reduction $\varrho_{gg'} : \mathbb{Z}_g \to \mathbb{Z}'_g \bmod g'$ is defined and is a ring homomorphism.

Again, let $D \in \mathrm{SU}n$, such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. Let $Q_L^{[h_D^{\mathrm{S}}]} = Q_L \times_{\mathrm{SU}J} \mathrm{SU}J'$ denote the $\mathrm{SU}J'$-subbundle of P associated to Q_L by virtue of the homomorphism $h_D^{\mathrm{S}} : \mathrm{SU}J \to \mathrm{SU}J'$.

Lemma 3.2. *The characteristic classes of* $Q_L^{[h_D^{\mathrm{S}}]}$ *are*

$$\alpha_{J'}(Q_L^{[h_D^{\mathrm{S}}]}) = E_{\Delta(D)}(\alpha), \tag{29}$$

$$\xi_{J'}(Q_L^{[h_D^{\mathrm{S}}]}) = \varrho_{gg'}(\xi), \tag{30}$$

Proof. The classifying map of $Q_L^{[h_D^{\mathrm{S}}]}$ is

$$f_{Q_L^{[h_D^{\mathrm{S}}]}} = \mathrm{B}h_D^{\mathrm{S}} \circ f_{Q_L}. \tag{31}$$

Hence, according to (2)

$$\begin{aligned} \alpha_{J',i'}(Q_L^{[h_D^{\mathrm{S}}]}) &= (f_{Q_L^{[h_D^{\mathrm{S}}]}})^*((\mathrm{B}j_{j'})^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}}) \\ &= (f_{Q_L})^*(\mathrm{B}h_D^{\mathrm{S}})^*((\mathrm{B}j_{j'})^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}}) \\ &= (f_{Q_L})^*(\mathrm{B}j_J)^*(\mathrm{B}h_D^{\mathrm{U}})^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}}. \end{aligned} \tag{32}$$

In order to calculate $(\mathrm{B}h_D^{\mathrm{U}})^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}}$, consider the homomorphisms

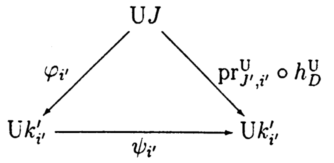
$$\mathrm{pr}^{\mathrm{M}}_{J',i'} \circ h_D^{\mathrm{M}} : \mathrm{M}_J(\mathbb{C}) \to \mathrm{M}_{k'_{i'}}(\mathbb{C}), \tag{33}$$

$$\mathrm{pr}^{\mathrm{U}}_{J',i'} \circ h_D^{\mathrm{U}} : \mathrm{U}J \to \mathrm{U}_{k'_{i'}}. \tag{34}$$

Since the image of (33) is a unital $*$-subalgebra of $\mathrm{M}_{k'_{i'}}(\mathbb{C})$, the image of (34) is a Howe subgroup of $\mathrm{U}k'_{i'}$. Hence, the latter is conjugate to $\mathrm{U}J^{(i')}$ for some $J^{(i')} \in \mathrm{K}(k'_{i'})$. One can check that $J^{(i')}$ is obtained from the pair of sequences $((k_1, \dots, k_r), (\Delta_{i'1}, \dots, \Delta_{i'r}))$ by deleting all pairs of entries k_i, $\Delta_{i'i}$ for which $\Delta_{i'i} = 0$. On the other hand, $\mathrm{U}J^{(i')}$ is the image of the homomorphism

$$\varphi_{i'} : \mathrm{U}J \xrightarrow{d_r} \prod_{i=1}^{r} \mathrm{U}J \xrightarrow{\prod_{i=1}^{r} \mathrm{pr}^{\mathrm{U}}_{J,i}} \prod_{i=1}^{r} \mathrm{U}k_i \xrightarrow{\prod_{i=1}^{r} d_{\Delta(D)_{i'i}}} \prod_{i=1}^{r} \left(\prod_{j=1}^{\Delta(D)_{i'i}} \mathrm{U}k_i \right) \xrightarrow{\iota_{i'}} \mathrm{U}k'_{i'}. \tag{35}$$

Here d_l denotes diagonal embedding into the l-fold product, where for $l = 0$ this product is assumed to reduce to $\{\mathbb{1}\}$, and $\iota_{i'}$ is a standard blockwise embedding. Having conjugate images, the homomorphisms (34) and (35) are conjugate themselves [7], i.e., there exists an inner automorphism $\psi_{i'}$ of $\mathrm{U}k'_{i'}$ such that the following diagram commutes:



(36)

Since $\mathrm{U}k'_{i'}$ is connected, $\mathrm{B}\psi_{i'}$ is null-homotopic. Thus, on the level of cohomology

$$(\mathrm{B}h^{\mathrm{U}}_D)^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}} = (\mathrm{B}\varphi_{i'})^*\gamma_{\mathrm{U}k'_{i'}}. \tag{37}$$

From the decomposition (35), one derives

$$(\mathrm{B}\varphi_{i'})^*\gamma_{\mathrm{U}k'_{i'}} = ((\mathrm{Bpr}^{\mathrm{U}}_{J,1})^*\gamma_{\mathrm{U}k_1})^{\Delta(D)_{i'1}} \smile \cdots \smile ((\mathrm{Bpr}^{\mathrm{U}}_{J,r})^*\gamma_{\mathrm{U}k_r})^{\Delta(D)_{i'r}}, \tag{38}$$

see the proof of Lemma 5.12 in [13] for details. We remark that (38) is an analog of the Whitney sum formula. Using (7), from (37) and (38) we deduce

$$(\mathrm{B}h^{\mathrm{U}}_D)^*(\mathrm{Bpr}^{\mathrm{U}}_{J',i'})^*\gamma_{\mathrm{U}k'_{i'}} = E_{\Delta(D),i'}((\mathrm{Bpr}^{\mathrm{U}}_{J,1})^*\gamma_{\mathrm{U}k_1}, \dots, (\mathrm{Bpr}^{\mathrm{U}}_{J,r})^*\gamma_{\mathrm{U}k_r}). \tag{39}$$

Inserting (39) into (32) and using (2) and (13), we find

$$\begin{aligned} \alpha_{J',i'}(Q_L^{[h^{\mathrm{S}}_D]}) &= (f_{Q_L})^*(\mathrm{B}j_J)^*E_{\Delta(D),i'}((\mathrm{Bpr}^{\mathrm{U}}_{J,1})^*\gamma_{\mathrm{U}k_1}, \dots, (\mathrm{Bpr}^{\mathrm{U}}_{J,r})^*\gamma_{\mathrm{U}k_r}) \\ &= (f_{Q_L})^*E_{\Delta(D),i'}((\mathrm{B}j_J)^*(\mathrm{Bpr}^{\mathrm{U}}_{J,1})^*\gamma_{\mathrm{U}k_1}, \dots, (\mathrm{B}j_J)^*(\mathrm{Bpr}^{\mathrm{U}}_{J,r})^*\gamma_{\mathrm{U}k_r}) \\ &= E_{\Delta(D),i'}(\alpha_J(Q_L)) = E_{\Delta(D),i'}(\alpha). \end{aligned}$$

This proves (29). Now consider (30), using (6) and (31), we compute

$$\begin{aligned} \beta_{g'}(\xi_{J'}(Q_L^{[h^{\mathrm{S}}_D]})) &= \beta_{g'}(f_{Q_L^{[h^{\mathrm{S}}_D]}})^*((\mathrm{B}\lambda^{\mathrm{S}}_{J'})^*\beta_{g'}^{-1}(\mathrm{B}j_{g'})^*\gamma^{(2)}_{\mathrm{U}1}) \\ &= (f_{Q_L^{[h^{\mathrm{S}}_D]}})^*(\mathrm{B}\lambda^{\mathrm{S}}_{J'})^*(\mathrm{B}j_{g'})^*\gamma^{(2)}_{\mathrm{U}1} \\ &= (f_{Q_L})^*(\mathrm{B}h^{\mathrm{S}}_D)^*(\mathrm{B}\lambda^{\mathrm{S}}_{J'})^*(\mathrm{B}j_{g'})^*\gamma^{(2)}_{\mathrm{U}1}. \end{aligned} \tag{40}$$

Let l be such that $g = lg'$. The following relation will be proved afterwards:

$$j_{g'} \circ \lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_{D} = p_l \circ j_g \circ \lambda^{\mathrm{S}}_{J}. \tag{41}$$

Inserting (41) into (40) yields

$$\beta_{g'}(\xi_{J'}(Q_L^{[h^{\mathrm{S}}_D]})) = (f_{Q_L})^*(\mathrm{B}\lambda^{\mathrm{S}}_J)^*(\mathrm{B}j_g)^*(\mathrm{B}p_l)^*\gamma^{(2)}_{\mathrm{U1}}. \tag{42}$$

It is easily seen that $(p_l)_* : \pi_1(\mathrm{U1}) \to \pi_1(\mathrm{U1})$ is multiplication by l. Therefore,

$$(\mathrm{B}p_l)^*\gamma^{(2)}_{\mathrm{U1}} = l\gamma^{(2)}_{\mathrm{U1}}. \tag{43}$$

Then (42) becomes

$$\begin{aligned}\beta_{g'}(\xi_{J'}(Q_L^{[h^{\mathrm{S}}_D]})) &= l(f_{Q_L})^*(\mathrm{B}\lambda^{\mathrm{S}}_J)^*(\mathrm{B}j_g)^*\gamma^{(2)}_{\mathrm{U1}} = l\beta_g(f_{Q_L})^*((\mathrm{B}\lambda^{\mathrm{S}}_J)^*\beta_g^{-1}(\mathrm{B}j_g)^*\gamma^{(2)}_{\mathrm{U1}}) \\ &= l\beta_g(\xi_J(Q_L)) = l\beta_g(\xi),\end{aligned} \tag{44}$$

where for the last two equalities, we have used (6) and (14), respectively. As a direct consequence of the definition of the Bockstein homomorphism, one has

$$l\beta_g = \beta_{g'}\varrho_{gg'}. \tag{45}$$

Thus, (44) yields

$$\beta_{g'}(\xi_{J'}(Q_L^{[h^{\mathrm{S}}_D]})) = \beta_{g'}\varrho_{gg'}(\xi). \tag{46}$$

Consider the following portion of the long exact sequence of coefficient homomorphisms which is induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{g'} \to 0$, see [5, Chapter IV and Section 5]:

$$\cdots \to H^1(\mathrm{BSU}J, \mathbb{Z}) \to H^1(\mathrm{BSU}J, \mathbb{Z}_{g'}) \xrightarrow{\beta_{g'}} H^2(\mathrm{BSU}J, \mathbb{Z}) \to \cdots.$$

Since $H^1(\mathrm{BSU}J, \mathbb{Z}) = 0$, see [13, Corollary 5.8], $\beta_{g'}$ is injective here. Hence (46) implies (30). It remains to prove the relation (41). According to (3) and (5), for any $\mathrm{B} \in \mathrm{SU}J$

$$\begin{aligned}j_{g'} \circ \lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_D(B) &= \lambda^{\mathrm{U}}_{J'} \circ j_J \circ h^{\mathrm{S}}_D(B) = \lambda^{\mathrm{U}}_{J'} \circ h^{\mathrm{U}}_D \circ j_J(B) \\ &= \prod_{i'=1}^{r'} p_{\tilde m'_{i'}} \circ \det\nolimits_{\mathrm{U}k'_{i'}} \circ \mathrm{pr}^{\mathrm{U}}_{J',i'} \circ h^{\mathrm{U}}_D \circ j_J(B).\end{aligned} \tag{47}$$

Using (36) to replace $\mathrm{pr}^{\mathrm{U}}_{J',i'} \circ h^{\mathrm{U}}_D$ and taking into account that an inner automorphism does not change the determinant, (47) yields

$$j_{g'} \circ \lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_D(B) = \prod_{i'=1}^{r'} p_{\tilde m'_{i'}} \circ \det\nolimits_{\mathrm{U}k'_{i'}} \circ \varphi_{i'} \circ j_J(B). \tag{48}$$

By construction of $\varphi_{i'}$, see (35), for any $C \in \mathrm{U}J$,

$$\det\nolimits_{\mathrm{U}k'_{i'}} \circ \varphi_{i'}(C) = \prod_{i=1}^{r} p_{\Delta(D)_{i'i}} \circ \det\nolimits_{\mathrm{U}k_i} \circ \mathrm{pr}^{\mathrm{U}}_{J,i}(C).$$

Thus, (48) becomes

$$\begin{aligned} j_{g'} \circ \lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_{D}(B) &= \prod_{i'=1}^{r'} \prod_{i=1}^{r} p_{\tilde{m}'_{i'}} \circ p_{\Delta(D)_{i'i}} \circ \det\nolimits_{\mathrm{U}k_i} \circ \mathrm{pr}^{\mathrm{U}}_{J,i} \circ j_J(B) \\ &= \prod_{i=1}^{r} p_{(\sum_{i'=1}^{r'} \tilde{m}'_{i'} \Delta(D)_{i'i})} \circ \det\nolimits_{\mathrm{U}k_i} \circ \mathrm{pr}^{\mathrm{U}}_{J,i} \circ j_J(B). \end{aligned} \tag{49}$$

Due to (22), $g' \sum_{i'=1}^{r'} \tilde{m}'_{i'} \Delta(D)_{i'i} = \sum_{i'=1}^{r'} m'_{i'} \Delta(D)_{i'i} = m_i = g\tilde{m}_i$, hence

$$\sum_{i'=1}^{r'} \tilde{m}'_{i'} \Delta(D)_{i'i} = l\tilde{m}_i, \quad i = 1, \ldots, r. \tag{50}$$

Consequently, (49) implies

$$\begin{aligned} j_{g'} \circ \lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_{D}(B) &= \prod_{i=1}^{r} p_{l\tilde{m}_i} \circ \det\nolimits_{\mathrm{U}k_i} \circ \mathrm{pr}^{\mathrm{U}}_{J,i} \circ j_J(B) \\ &= p_l \left(\prod_{i=1}^{r} p_{\tilde{m}_i} \circ \det\nolimits_{\mathrm{U}k_i} \circ \mathrm{pr}^{\mathrm{U}}_{J,i} \circ j_J(B) \right) \\ &= p_l \circ \lambda^{\mathrm{U}}_{J} \circ j_J(B) = p_l \circ j_g \circ \lambda^{\mathrm{S}}_{J}(B), \end{aligned}$$

where the last two equalities are due to (3) and (5), respectively. This proves (41) and, therefore, concludes the proof of the lemma. □

Lemma 3.3. *Let $D \in \mathrm{SU}n$, such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. Then $Q_L \cdot D$ is a reduction of $Q_L^{[h^{\mathrm{S}}_D]}$ to the structure group $D^{-1}\mathrm{SU}JD$.*

Proof. Define a map $\varphi : Q_L \cdot D \to Q_L^{[h^{\mathrm{S}}_D]}$, $q \cdot D \mapsto [(q, \mathbb{1})]$. This map is obviously smooth. To check equivariance, let $C \in \mathrm{SU}J$, then

$$\begin{aligned} \varphi((q \cdot D) \cdot D^{-1}CD) &= \varphi((q \cdot C) \cdot D) = [(q \cdot C, \mathbb{1})] = [(q, h^{\mathrm{S}}_D(C))] \\ &= [(q, \mathbb{1})] \cdot h^{\mathrm{S}}_D(C) = [(q, \mathbb{1})] \cdot D^{-1}CD. \end{aligned}$$

This proves the lemma. □

Theorem 3.4. *Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ be elements of $\mathrm{K}(P)$. Then $[Q_L] \leq [Q_{L'}]$ if and only if*

(a) *g' divides g and there holds $\xi' = \varrho_{gg'}(\xi)$,*
(b) *there exists $\Delta \in \mathrm{M}_{r',r}(\mathbb{N})$ such that*

$$\Delta \mathbf{k} = \mathbf{k}', \tag{51}$$

$$\mathbf{m} = \mathbf{m}'\Delta, \tag{52}$$

$$E_\Delta(\alpha) = \alpha'. \tag{53}$$

Proof. To begin with, assume $[Q_L] \leq [Q_{L'}]$. Then there exists $D \in \mathrm{SU}n$ such that $Q_L \cdot D \subseteq Q_{L'}$. Since $Q_L \cdot D$ has structure group $D^{-1}\mathrm{SU}JD$, $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$. As a consequence, the homomorphism h^{S}_D and the inclusion matrix $\Delta(D)$ exist. Due to Lemma 3.1, $\Delta(D) \in \mathrm{N}(J, J')$, hence it obeys (51) and (52). The latter equation implies, in particular, that g' divides g. Moreover, by construction, $Q_{L'}$ can be reduced to $Q_L \cdot D$. According to Lemma 3.3, so can the $\mathrm{SU}J'$-bundle $Q_L^{[h^{\mathrm{S}}_D]}$. Since $Q_{L'}$ and $Q_L^{[h^{\mathrm{S}}_D]}$ have the same structure group, it follows $Q_{L'} \cong Q_L^{[h^{\mathrm{S}}_D]}$. Then Lemma 3.2 yields

$$\alpha' = \alpha_{J'}(Q_{L'}) = \alpha_{J'}(Q_L^{[h^{\mathrm{S}}_D]}) = E_{\Delta(D)}(\alpha).$$

Thus, $\Delta(D)$ satisfies (53). By an analogous argument, we finally find $\xi' = \varrho_{gg'}(\xi)$.

Conversely, assume that assertions (a) and (b) hold. Then, due to Lemma 3.1, there exists $D \in \mathrm{SU}n$ such that $D^{-1}\mathrm{SU}JD \subseteq \mathrm{SU}J'$ and $\Delta(D) = \Delta$. Consider the $\mathrm{SU}J'$-bundle $Q_L^{[h^{\mathrm{S}}_D]}$ associated to Q_L. Due to Lemma 3.2 and (53)

$$\alpha_{J'}(Q_L^{[h^{\mathrm{S}}_D]}) = E_{\Delta}(\alpha) = \alpha' = \alpha_{J'}(Q_{L'}).$$

Analogously, we obtain $\xi_{J'}(Q_L^{[h^{\mathrm{S}}_D]}) = \xi_{J'}(Q_{L'})$. Hence, $Q_{L'}$ and $Q_L^{[h^{\mathrm{S}}_D]}$ are isomorphic. Then Lemma 3.3 implies $Q_L \cdot D \subseteq Q_{L'}$, up to isomorphy (which is sufficient). It follows $[Q_L] \leq [Q_{L'}]$. □

Let $L, L' \in \mathrm{K}(P)$. If condition (a) of Theorem 3.4 holds, we define $\mathrm{N}(L, L')$ to be the set of solutions of the system of equations (51)–(53). If this condition does not hold, we define $\mathrm{N}(L, L') = \emptyset$. In order to be able to argue entirely on the level of $\hat{\mathrm{K}}(P)$, we define a partial ordering on $\hat{\mathrm{K}}(P)$ as the image of the natural partial ordering of $\mathrm{Howe}_*(P)$ under the bijection defined by the collection of characteristic classes α_J, ξ_J, $J \in \mathrm{K}(n)$. According to Theorem 3.4, the partial ordering so defined can be characterized as follows.

Corollary 3.5. *Let $\kappa, \kappa' \in \hat{\mathrm{K}}(P)$, then the following assertions are equivalent*:

(a) $\kappa \leq \kappa'$;
(b) *there exist representatives L, L' of κ, κ', respectively, such that* $\mathrm{N}(L, L')$ *is non-empty*;
(c) *for any two representatives L, L' of κ, κ', respectively,* $\mathrm{N}(L, L')$ *is non-empty*.

Proof.

(a) ⇒ (c): Let L, L' be given. By assumption, $[Q_L] \leq [Q_{L'}]$, then Theorem 3.4 implies that $\mathrm{N}(L, L')$ is non-empty.
(c) ⇒ (b): Obvious.
(b) ⇒ (a): Let L, L' be the representatives provided by assertion (b). Since $\mathrm{N}(L, L')$ is non-empty, assertions (a) and (b) of Theorem 3.4 hold. It follows that the subbundles Q_L and $Q_{L'}$ obey $[Q_L] \leq [Q_{L'}]$, hence, $\kappa \leq \kappa'$. □

Example. Let $P = M \times \mathrm{SU}4$. Consider elements $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ of $\mathrm{K}(P)$, where $J = ((1, 1), (2, 2))$ and $J' = ((2, 2), (1, 1))$. We remark that the subgroup

$\mathrm{SU}J \subseteq \mathrm{SU4}$ has connected components

$$\left\{ \begin{pmatrix} z\mathbb{1}_2 & 0 \\ 0 & z^{-1}\mathbb{1}_2 \end{pmatrix} \middle| z \in \mathrm{U1} \right\}, \qquad \left\{ \begin{pmatrix} z\mathbb{1}_2 & 0 \\ 0 & -z^{-1}\mathbb{1}_2 \end{pmatrix} \middle| z \in \mathrm{U1} \right\},$$

hence is isomorphic to the direct product $\mathbb{Z}_2 \times \mathrm{U1}$. The subgroup $\mathrm{SU}J'$ can be parameterized as follows:

$$\mathrm{SU}J' = \left\{ \begin{pmatrix} zA & 0 \\ 0 & z^{-1}B \end{pmatrix} \middle| z \in \mathrm{U1},\, A,\, B \in \mathrm{SU2} \right\}.$$

Thus, it is isomorphic to the direct product $\mathrm{U1} \times \mathrm{SU2} \times \mathrm{SU2}$.

In order to find out whether $[Q_L] \leq [Q_{L'}]$, we are going to determine $\mathrm{N}(L, L')$. Condition (a) of Theorem 3.4 is obviously satisfied. Thus, we can proceed as follows: first, we solve Eqs. (51) and (52), i.e., we derive $\mathrm{N}(J, J')$. Then, for all $\Delta \in \mathrm{N}(J, J')$, we compute $E_\Delta(\alpha)$ and compare the result with α'. Eqs. (51) and (52) read

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad (\,1 \quad 1\,) \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = (\,2 \quad 2\,).$$

We extract the equations

$$\Delta_{11} + \Delta_{12} = 2, \qquad \Delta_{21} + \Delta_{22} = 2, \qquad \Delta_{11} + \Delta_{21} = 2, \qquad \Delta_{12} + \Delta_{22} = 2.$$

The solutions are

$$\Delta^a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Delta^b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Delta^c = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{54}$$

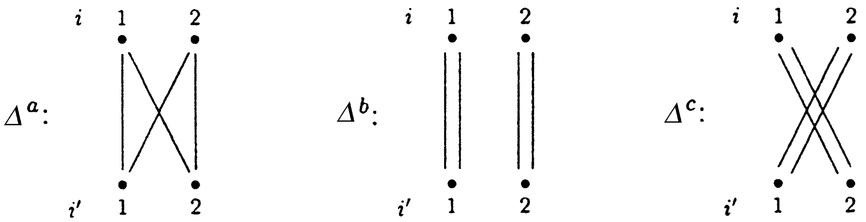
For $\alpha = (\alpha_1, \alpha_2)$, they yield

$$E_{\Delta^a}(\alpha) = (\alpha_1 \smile \alpha_2, \alpha_1, \smile \alpha_2), \qquad E_{\Delta^b}(\alpha) = (\alpha_1 \smile \alpha_1, \alpha_2, \smile \alpha_2),$$
$$E_{\Delta^c}(\alpha) = (\alpha_2 \smile \alpha_2, \alpha_1, \smile \alpha_1).$$

Thus, $\mathrm{N}(L, L') \neq \emptyset$, i.e., $[Q_L] \leq [Q_{L'}]$, if and only if α' coincides with one of the elements $E_{\Delta^a}(\alpha)$, $E_{\Delta^b}(\alpha)$, or $E_{\Delta^c}(\alpha)$ listed above.

4. Bratteli diagrams

Any $\Delta \in \mathrm{M}_{r',r}(\mathbb{N})$ can be visualized by a diagram consisting of a series of upper vertices, labeled by $i = 1, \ldots, r$, and a series of lower vertices, labeled by $i' = 1, \ldots, r'$. For each combination of i and i', the corresponding vertices are connected by $\Delta_{i'i}$ edges. For example, the matrices Δ^a, Δ^b and Δ^c in (54) give rise to the following diagrams:



The diagrams associated in this way to the elements of $\mathrm{N}(J, J')$, where $J, J' \in \mathrm{K}(n)$ are special cases of the so-called *Bratteli diagrams* [4]. The latter have, in general, several stages picturing the subsequent inclusion matrices associated to an ascending sequence of finite-dimensional von Neumann algebras $\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \mathbf{A}_3 \subseteq \cdots$. For this reason, we refer to the diagram associated to $\Delta \in \mathrm{N}(J, J')$ as the Bratteli diagram of Δ. We remark that due to Eq. (51), Δ cannot have a zero row. Due to (52), it cannot have a zero column either. Accordingly, each vertex of the Bratteli diagram of Δ is cut by at least one edge.

Let $L = (J; \alpha, \xi)$ and $L' = (J'; \alpha'\xi')$ be elements of $\mathrm{K}(P)$. In terms of the Bratteli diagram of the variable Δ, Eqs. (51)–(53) can be rewritten as follows:

$$k'_{i'} = \sum_{i=1}^{r} \sum_{\text{edges from } i \text{ to } i'} k_i, \quad i' = 1, \ldots, r', \tag{55}$$

$$m_i = \sum_{i'=1}^{r'} \sum_{\text{edges from } i \text{ to } i'} m'_{i'}, \quad i = 1, \ldots, r, \tag{56}$$

$$\alpha'_{i'} = \mathop{\smile}_{i=1}^{r} \mathop{\smile}_{\text{edges from } i \text{ to } i'} \alpha_i, \quad i' = 1, \ldots, r'. \tag{57}$$

The main use of Bratteli diagrams is to simplify calculations as, for instance, solving the equations determining $\mathrm{N}(L, L')$. Furthermore, some of the arguments in the sequel are easier to formulate on the level of these diagrams than on the level of the corresponding matrices.

5. Direct successors

In this section, we are going to derive a characterization of direct successor relations in $\hat{\mathrm{K}}(P)$ and to formulate operations that generate the direct successors of any given element of $\hat{\mathrm{K}}(P)$.

5.1. *The level of an inclusion matrix*

Let $J, J' \in \mathrm{K}(n)$. For any $\Delta \in \mathrm{N}(J, J')$, we define the level of Δ to be the integer

$$\ell(\Delta) = 2 \sum_{i=1}^{r} \sum_{i'=1}^{r'} \Delta_{i'i} - (r + r'). \tag{58}$$

Using the quantities

$$\ell_i^+(\Delta) = \left(\sum_{i'=1}^{r'} \Delta_{i'i}\right) - 1, \quad i = 1, \ldots, r, \tag{59}$$

$$\ell_{i'}^-(\Delta) = \left(\sum_{i=1}^{r} \Delta_{i'i}\right) - 1, \quad i' = 1, \ldots, r', \tag{60}$$

we can write

$$\ell(\Delta) = \sum_{i=1}^{r} \ell_i^+(\Delta) + \sum_{i'=1}^{r'} \ell_{i'}^-(\Delta). \tag{61}$$

Due to (51) and (52), each row and each column of Δ contain at least one non-zero entry. It follows that $\ell_i^+(\Delta), \ell_{i'}^-(\Delta) \geq 0$. Hence, due to (61), $\ell(\Delta) \geq 0$.

As for the interpretation, $\ell(\Delta)$ measures, in a sense, how much J' deviates from J (up to permutations). On the level of the Bratteli diagram of Δ, $\ell(\Delta)$ is twice the number of edges minus the number of vertices, whereas $\ell_i^+(\Delta)$ and $\ell_{i'}^-(\Delta)$ count the edges at the vertices i and i', respectively, minus the obligatory one edge per vertex.

For later use, we note the following formulae, which follow immediately from (61)

$$\ell(\Delta) = 2\sum_{i=1}^{r} \ell_i^+(\Delta) + r - r' = 2\sum_{i'=1}^{r'} \ell_{i'}^-(\Delta) + r' - r. \tag{62}$$

5.2. *Lemmata about the level*

Lemma 5.1. *Let* $L, L', L'' \in \mathrm{K}(P)$ *and let* $\Delta \in \mathrm{N}(L, L')$, $\Delta' \in \mathrm{N}(L', L'')$. *Then* $\Delta'\Delta \in \mathrm{N}(L, L'')$ *and*

$$\ell(\Delta'\Delta) \geq \ell(\Delta') + \ell(\Delta). \tag{63}$$

Moreover, $\ell(\Delta') = 0$ *or* $\ell(\Delta) = 0$ *imply equality in* (63).

Proof. Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ and $L'' = (J''; \alpha'', \xi'')$. By the assumption that $\mathrm{N}(L, L')$ and $\mathrm{N}(L', L'')$ be non-empty, g' divides g and g'' divides g', hence g'' divides g. Also by this assumption, $\xi' = \varrho_{gg'}(\xi)$ and $\xi'' = \varrho_{g'g''}(\xi')$, hence $\varrho_{gg''}(\xi) = \varrho_{g'g''} \circ \varrho_{gg'}(\xi) = \varrho_{g'g''}(\xi') = \xi''$. Moreover, one can check that $\Delta'\Delta$ obeys Eqs. (51)–(53), where for the last one, (18) has to be used.

To prove (63), using (58)–(60), we compute

$$\begin{aligned} 2\sum_{i'=1}^{r'} \ell_{i'}^+(\Delta')\ell_{i'}^-(\Delta) &= 2\sum_{i'=1} \left(\left(\sum_{i''=1}^{r''} \Delta'_{i''i'}\right) - 1\right)\left(\left(\sum_{i=1}^{r} \Delta_{i'i}\right) - 1\right) \\ &= 2\left(\sum_{i''=1}^{r''}\sum_{i'=1}^{r'}\sum_{i=1}^{r} \Delta'_{i''i'}\Delta_{i'i} - \sum_{i'=1}^{r'}\sum_{i=1}^{r} \Delta_{i'i} - \sum_{i''=1}^{r''}\sum_{i'=1}^{r'} \Delta'_{i''i'} + r'\right) \\ &= \ell(\Delta'\Delta) - \ell(\Delta) - \ell(\Delta'). \end{aligned} \tag{64}$$

Since the l.h.s. of (64) is non-negative, this yields (63). Moreover, if $\ell(\Delta) = 0$ or $\ell(\Delta') = 0$, then due to (61), $\ell^-_{i'}(\Delta) = 0$ or $\ell^+_{i'}(\Delta') = 0$, respectively, for all i'. Hence, the l.h.s. of (64) vanishes, so that equality holds in (63). □

Lemma 5.2. *Let $L, L' \in \mathrm{K}(P)$ and let $l = 0$ or 1. If $\mathrm{N}(L, L')$ contains an element of level l then all its elements have level l.*

Proof. Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ and let $\Delta \in \mathrm{N}(L, L')$. Due to (51) and (59)

$$\sum_{i=1}^{r} k_i \ell_i^+(\Delta) = \sum_{i=1}^{r} k_i \left(\left(\sum_{i'=1}^{r'} \Delta_{i'i} \right) - 1 \right) = \sum_{i'=1}^{r'} k'_{i'} - \sum_{i=1}^{r} k_i. \tag{65}$$

Since $k_i > 0$ and $\ell_i^+(\Delta) \geq 0$ for all i, (65) implies

$$\ell_i^+(\Delta) = 0 \quad \forall i \Leftrightarrow \sum_{i'=1}^{r'} k'_{i'} - \sum_{i=1}^{r} k_i = 0. \tag{66}$$

By a similar argument, we find

$$\ell_{i'}^-(\Delta) \quad \forall i = 0 \Leftrightarrow \sum_{i=1}^{r} m_i - \sum_{i'=1}^{r'} m'_{i'} = 0. \tag{67}$$

Now assume that $\ell(\Delta) = l$, where $l = 0$ or 1. Then at most one of the integers $\ell_i^+(\Delta)$ or $\ell_{i'}^-(\Delta)$ can be non-zero. Thus, (66) or (67) holds. In either case, the assertion holds for any $\Delta' \in \mathrm{N}(L, L')$. Then (62) implies $\ell(\Delta') = \ell(\Delta) = l$. □

Remarks.

(1) The proof of Lemma 5.2 shows that the lemma still holds if one replaces $\mathrm{N}(L, L')$ by $\mathrm{N}(J, J')$, for any $J, J' \in \mathrm{K}(n)$.
(2) In general, the level function ℓ may not be constant on the sets $\mathrm{N}(L, L')$. For example, let P be the trivial SU8-bundle over M and let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$ be given by $J = ((1, 2), (4, 2)), \alpha = 1, \xi = 0$ and $J' = ((4, 2), (1, 2)), \alpha' = 1, \xi' = 0$. Obviously, $(\alpha, \xi) \in \mathrm{K}(P)_J$ and $(\alpha', \xi') \in \mathrm{K}(P)_{J'}$. One can check that $\mathrm{N}(L, L')$ contains the following two inclusion matrices:

$$\Delta = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta' = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

One has $\ell(\Delta) = 6$ and $\ell(\Delta') = 4$.

Lemma 5.3. *Let $L, L' \in \mathrm{K}(P)$. The following assertions are equivalent*:

(a) *L and L' are equivalent.*
(b) *$\mathrm{N}(L, L')$ contains an element of level* 0.
(c) *$\mathrm{N}(L, L')$ is non-empty and all of its elements have level* 0.

Proof. Due to Lemma 5.2, (b) ⇔ (c). Hence, it suffices to prove (a) ⇔ (b). Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi')$. First, assume that there exist $\Delta \in \mathrm{N}(L, L')$ such that $\ell(\Delta) = 0$. Then $\ell_i^+(\Delta) = 0$ for all i and $\ell_{i'}^-$ for all i'. That means, each row and each column contains exactly one non-zero entry and this entry has value 1. It follows that Δ is square, i.e., $r' = r$, and that there exists a permutation σ of $1, \ldots, r$ such that

$$\Delta_{i'i} = \delta_{\sigma(i')i}, \quad i', i = 1, \ldots, r. \tag{68}$$

As an immediate consequence

$$\Delta \mathbf{k} = \sigma \mathbf{k}, \qquad \mathbf{m}' \Delta = \sigma^{-1} \mathbf{m}', \qquad E_\Delta(\alpha) = \sigma \alpha. \tag{69}$$

Since $\Delta \in \mathrm{N}(L, L')$, (69) implies $J' = \sigma J$, $\alpha' = \sigma \alpha$, and $\xi' = \varrho_{gg'}(\xi)$. In particular, $\mathbf{m}' = \sigma \mathbf{m}$, hence $g = g'$. It follows $\xi' = \xi$, thus, L and L' are equivalent.

Conversely, assume that $\xi' = \xi$ and that there exist a permutation σ of $1, \ldots, r$ such that $J' = \sigma J$ and $\alpha' = \sigma \alpha$. Since, in particular, $\mathbf{m}' = \sigma \mathbf{m}$, g' and g coincide. Thus, trivially, g' divides g and $\xi' = \varrho_{gg'}(\xi)$. Hence, if we find a solution Δ of Eqs. (51)–(53) then $\Delta \in \mathrm{N}(L, L')$. Due to (69), such a solution is given by the matrix (68). By construction, it has level 0. □

5.3. *Splitting and merging*

Let $L = (J; \alpha, \xi) \in \mathrm{K}(P)$. In this section, we are going to formulate operations that create new elements of $\mathrm{K}(P)$ out of L. These operations will be used to prove a decomposition lemma in Section 5.4 and, later on, to generate direct successors.

5.3.1. *Splitting*

Choose $1 \leq i_0 \leq r$ such that $m_{i_0} \neq 1$. Choose a decomposition $m_{i_0} = m_{i_0,1} + m_{i_0,2}$ with strictly positive integers $m_{i_0,1}, m_{i_0,2}$. Define sequences of length $(r+1)$

$$\mathbf{k}^\circ = (k_1, \ldots, k_{i_0-1}, k_{i_0}, k_{i_0}, k_{i_0+1}, \ldots, k_r), \tag{70}$$

$$\mathbf{m}^\circ = (m_1, \ldots, m_{i_0-1}, m_{i_0,1}, m_{i_0,2}, m_{i_0+1}, \ldots, m_r). \tag{71}$$

$$\alpha^\circ = (\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0}, \alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_r). \tag{72}$$

Since the greatest common divisor g° of $\mathbf{m}^\circ$ divides g, we can furthermore define

$$\xi^\circ = \varrho_{gg^\circ}(\varphi). \tag{73}$$

Denote $J^\circ = (\mathbf{k}^\circ, \mathbf{m}^\circ)$ and $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$.

We claim that $L^\circ \in \mathrm{K}(P)$. It is easily seen that $\mathbf{m}^\circ \cdot \mathbf{k}^\circ = n$ and $\alpha \in H^{(J^\circ)}(M, \mathbb{Z})$. Consequently, it suffices to check that α° and ξ° obey Eqs. (10) and (11). First, consider (10). Let the integer l be such that $g = lg^\circ$. Using (45) and (17) as well as taking into account that (10) holds for α and ξ, we compute

$$\beta_{g^\circ}(\xi^\circ) = \beta_{g^\circ} \circ \varrho_{gg^\circ}(\xi) = l\beta_g(\xi) = lE^{(2)}_{\tilde{\mathbf{m}}}(\alpha) = E^{(2)}_{l\tilde{\mathbf{m}}}(\alpha).$$

Expanding the r.h.s. according to (15) yields

$$\begin{aligned}\beta_{g^\circ}(\xi^\circ) &= l\frac{m_1}{g}\alpha_1^{(2)} + \cdots + l\frac{m_{i_0}}{g}\alpha_{i_0}^{(2)} + \cdots + l\frac{m_r}{g}\alpha_r^{(2)} \\ &= \frac{m_1}{g^\circ}\alpha_1^{(2)} + \cdots + \frac{m_{i_0}}{g^\circ}\alpha_{i_0}^{(2)} + \cdots + \frac{m_r}{g^\circ}\alpha_r^{(2)} \\ &= \frac{m_1}{g^\circ}\alpha_1^{(2)} + \cdots + \frac{m_{i_0,1}}{g^\circ}\alpha_{i_0}^{(2)} + \frac{m_{i_0,2}}{g^\circ}\alpha_{i_0}^{(2)} + \cdots + \frac{m_r}{g^\circ}\alpha_r^{(2)} = E_{\tilde{\mathbf{m}}^\circ}(\alpha^\circ),\end{aligned}$$

where the penultimate equality is due to the fact that g° divides both $m_{i_0,1}$ and $m_{i_0,2}$. Now consider (11). Using commutativity of the cup product in even degree, one can check that $E_{\mathbf{m}^\circ}(\alpha^\circ) = E_{\mathbf{m}}(\alpha)$. Since (11) holds for α, it holds for α°. This proves $L^\circ \in \mathrm{K}(P)$. We say that L° arises from L by a splitting of the i_0th member.

5.3.2. Merging

Choose $1 \le i_1 \le i_2 \le r$ such that $m_{i_1} = m_{i_2}$. Define sequences of length $(r-1)$

$$\mathbf{k}^\circ = (k_1, \ldots, k_{i_1-1}, k_{i_1} + k_{i_2}, k_{i_1+1}, \ldots, \widehat{k_{i_2}}, \ldots, k_r), \tag{74}$$

$$\mathbf{m}^\circ = (m_1, \ldots, m_{i_1-1}, m_{i_1}, m_{i_1+1}, \ldots, \widehat{m_{i_2}}, \ldots, m_r), \tag{75}$$

$$\alpha^\circ = (\alpha_1, \ldots, \alpha_{i_1-1}, \alpha_{i_1} \smile \alpha_{i_2}, \alpha_{i_1+1}, \ldots, \widehat{\alpha_{i_2}}, \ldots, \alpha_r), \tag{76}$$

where $(\hat{\ })$ indicates that the entry is omitted, as well as

$$\xi^\circ = \xi. \tag{77}$$

Denote $J^\circ = (\mathbf{k}^\circ, \mathbf{m}^\circ)$ and $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$.

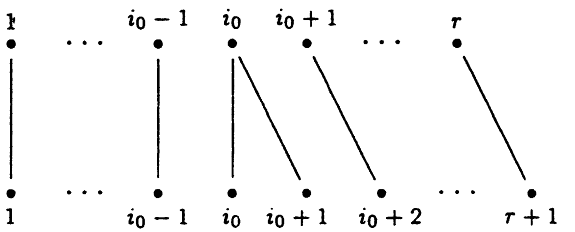
Let us show $L^\circ \in \mathrm{K}(P)$. As in the case of splitting, one can immediately verify that $\mathbf{m}^\circ \cdot \mathbf{k}^\circ = n$, $\alpha^\circ \in H^{(J^\circ)}(M, \mathbb{Z})$, and $E_{\mathbf{m}^\circ}(\alpha^\circ) = E_{\mathbf{m}}(\alpha)$. It follows that α° obeys Eq. (11).

Due to $g^\circ = g$, a similar calculation shows $E_{\tilde{\mathbf{m}}^\circ}(\alpha^\circ) = E_{\tilde{\mathbf{m}}}(\alpha)$. Since also $\beta_{g^\circ}(\xi^\circ) = \beta_g(\xi)$, we obtain $\beta_{g^\circ}(\xi^\circ) = E^{(2)}_{\tilde{\mathbf{m}}^\circ}(\alpha^\circ)$. Thus, $L^\circ \in \mathrm{K}(P)$.

We say that L° arises from L by merging the i_1th and the i_2th member.

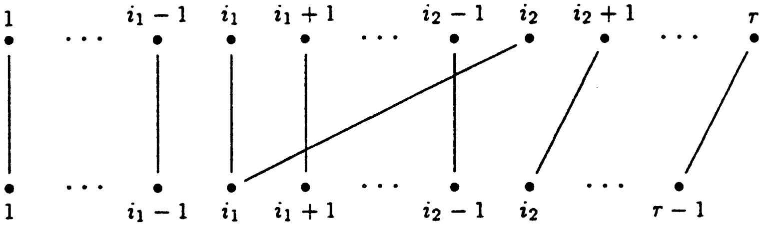
Remark. It may happen that for certain elements of $\mathrm{K}(P)$ no splittings or no mergings can be applied. Amongst these elements are, for example, those with $m_1 = \cdots = m_r = 1$ (no splitting) and those having pairwise distinct m_i (no merging).

Lemma 5.4. *Let $L, L^\circ \in \mathrm{K}(P)$. L° can be obtained from L by a splitting of the i_0th member if and only if $\mathrm{N}(L, L^\circ)$ contains an element with Bratteli diagram*



(78)

L° can be obtained from L by merging the i_1th and the i_2th member if and only if $\mathrm{N}(L, L^\circ)$ *contains an element with Bratteli diagram*



(79)

Proof. Assume $L = (J; \alpha, \xi)$, $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$. Since the proofs for the cases of splitting and merging are completely analogous, we only give the first one. First, assume that L° arises from L by a splitting of the i_0th member. Then, by construction, g° divides g and $\xi^\circ = \varrho_{gg^\circ}(\xi)$. Hence the matrix given by the Bratteli diagram (78) belongs to $\mathrm{N}(L, L^\circ)$ iff it satisfies Eqs. (51)–(53). By the help of Eqs. (55)–(57), this can be easily checked on diagram level. Conversely, assume that $\mathrm{N}(L, L^\circ)$ contains an element with Bratteli diagram (78). Then, in particular, condition (a) of Theorem 3.4 holds, i.e., g° divides g and $\xi^\circ = \varrho_{gg^\circ}(\xi)$. An inspection of (55)–(57) shows that $k^\circ_{i_0} = k^\circ_{i_0+1} = k_{i_0}$, $m^\circ_{i_0} + m^\circ_{i_0+1} = m_{i_0}$, and $\alpha^\circ_{i_0} = \alpha^\circ_{i_0+1} = \alpha_{i_0}$, whereas $k^\circ_i = k_i$, $m^\circ_i = m_i$, $\alpha^\circ_i = \alpha_i$ for $1 \le i < i_0$ and $k^\circ_{i+1} = k_i$, $m^\circ_{i+1} = m_i$, $\alpha^\circ_{i+1} = \alpha_i$ for $i_0 < i \le r$. Thus, L° is obtained from L by a splitting of the i_0th member according to the decomposition $m_{i_0} = m^\circ_{i_0} + m^\circ_{i_0+1}$. □

5.4. The decomposition lemma

Lemma 5.5. *Let* $L, L' \in \mathrm{K}(P)$ *and let* $\Delta \in \mathrm{N}(L, L')$. *If* $\ell(\Delta) \neq 0$ *then there exist* $L^\circ \in \mathrm{K}(P)$ *and* $\Delta^\circ \in \mathrm{N}(L, L^\circ)$, $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$ *such that* $\Delta = \Delta^{\circ\prime}\Delta^\circ$ *and* $\ell(\Delta^\circ) = 1$.

Proof. To begin with, assume that there exist i_0 such that $\ell^+_{i_0}(\Delta) > 0$. Choose i'_0 such that $\Delta_{i'_0 i_0} \neq 0$. We have the following estimate:

$$m_{i_0} - m'_{i'_0} = \sum_{i'=1}^{r'} m'_{i'}(\Delta_{i' i_0} - \delta_{i' i'_0}) \ge \sum_{i'=1}^{r'} (\Delta_{i' i_0} - \delta_{i' i'_0}) = \ell^+_{i_0}(\Delta) > 0.$$

This shows that $m_{i_0} = (m_{i_0} - m'_{i'_0}) + m'_{i'_0}$ is a decomposition into strictly positive integers. We define L° to be the element of $\mathrm{K}(P)$ obtained from L by the corresponding splitting operation. Furthermore, we define Δ° to be the $((r+1) \times r)$-matrix

$$\Delta^\circ = \left(\begin{array}{c|c} \mathbb{1}_{i_0} & 0 \\ \hline 0 \cdots 0\ 1 & 0 \cdots 0 \\ \hline 0 & \mathbb{1}_{r-i_0} \end{array}\right) \tag{80}$$

and $\Delta^{\circ\prime}$ to be the $(r' \times (r+1))$-matrix

$$\Delta^{\circ\prime} = \left(\begin{array}{ccc|c|ccc} \Delta_{11} & \cdots & \Delta_{1i_0} & 0 & \Delta_{1\ i_0+1} & \cdots & \Delta_{1r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \Delta_{i'_0-1\ i_0} & 0 & \vdots & & \vdots \\ \Delta_{i'_0\ 1} & \cdots & \Delta_{i'_0 i_0} - 1 & 1 & \Delta_{i'_0\ i_0+1} & \cdots & \Delta_{i'_0\ r} \\ \vdots & & \Delta_{i'_0+1\ i_0} & 0 & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{r'1} & \cdots & \Delta_{r'\ i_0} & 0 & \Delta_{r'\ i_0+1} & \cdots & \Delta_{r'\ r} \end{array}\right) \tag{81}$$

We notice that Δ° has Bratteli diagram (78). Hence, due to Lemma 5.4, $\Delta^\circ \in \mathrm{N}(L, L^\circ)$. From the diagram, we read off that $\ell(\Delta^\circ) = 1$. Moreover, by means of a direct computation using (80) and (81), one can check that $\Delta^{\circ\prime}\Delta^\circ = \Delta$. Thus, it remains to prove that $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$. This amounts to the following items:

(a) g' divides g°: We recall from (71) that

$$\mathbf{m}^\circ = (m_1, \ldots, m_{i_0-1}, m_{i_0} - m'_{i'_0}, m'_{i'_0}, m_{i_0+1}, \ldots, m_r). \tag{82}$$

By assumption, g' divides g, hence all the m_i. By definition, it also divides $m'_{i'_0}$.

(b) $\varrho_{g^\circ g'}(\xi^\circ) = \xi'$: According to (73), $\varrho_{g^\circ g'}(\xi^\circ) = \varrho_{g^\circ g'} \circ \varrho_{gg^\circ}(\xi) = \varrho_{gg'}(\xi) = \xi'$. Here the last equality holds by assumption.

(c) $\Delta^{\circ\prime}\mathbf{k}^\circ = \mathbf{k}'$: Using that $\Delta^\circ \in \mathrm{N}(L, L^\circ)$ and $\Delta \in \mathrm{N}(L, L')$, we compute $\Delta^{\circ\prime}\mathbf{k}^\circ = \Delta^{\circ\prime}\Delta^\circ\mathbf{k} = \Delta\mathbf{k} = \mathbf{k}'$.

(d) $\mathbf{m}'\Delta^{\circ\prime} = \mathbf{m}^\circ$: This has to be checked by a direct computation using (81) and (82).

(e) $E_{\Delta^{\circ\prime}}(\alpha^\circ) = \alpha'$: Using the same arguments as for (c), as well as (18), we obtain $E_{\Delta^{\circ\prime}}(\alpha^\circ) = E_{\Delta^{\circ\prime}} \circ E_{\Delta^\circ}(\alpha) = E_{\Delta^{\circ\prime}\Delta^\circ}(\alpha) = E_\Delta(\alpha) = \alpha'$.

This proves $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$.

Now assume that $\ell_i^+(\Delta) = 0$ for all i. Then in each column of Δ there exists exactly one non-zero entry, and this entry has value 1. On the other hand, since $\ell(\Delta) \neq 0$, there exists i'_0 such that $\ell^-_{i'_0}(\Delta) > 0$. This means, the row labeled by i'_0 has at least two entries of value 1. Therefore, we find two columns, labeled by $i_1 < i_2$ such that

$$\Delta_{i'i_k} = \begin{cases} 1, & i' = i'_0, \quad k = 1, 2, \\ 0 & \text{otherwise}, \quad k = 1, 2. \end{cases} \tag{83}$$

Then $m_{i_k} = \sum_{i'=1}^{r'} \Delta_{i'i_k} m'_{i'} = m'_{i'_0}$, $k = 1, 2$, hence $m_{i_1} = m_{i_2}$. Thus, we can define L° to be the element of $\mathrm{K}(P)$ obtained by merging the i_1th and the i_2th member of L.

Moreover, we define Δ° to be the $((r-1) \times r)$-matrix

$$\Delta^\circ = \left(\begin{array}{c|c|c|c|c} & 0 & & 0 & \\ \mathbb{1}_{i_1-1} & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\ & 0 & & 0 & \\ \hline 0 \ \cdots \ 0 & 1 & 0 \ \cdots \ 0 & 1 & 0 \ \cdots \ 0 \\ \hline & 0 & & 0 & \\ \mathbf{0} & \vdots & \mathbb{1}_{i_2-i_1-1} & \vdots & \mathbf{0} \\ & 0 & & 0 & \\ \hline & 0 & & 0 & \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbb{1}_{r-i_2} \\ & 0 & & 0 & \end{array}\right),$$

and $\Delta^{\circ\prime}$ to be the $(r' \times (r-1))$-matrix

$$\Delta^{\circ\prime} = \left(\begin{array}{ccc|c|ccc|ccc} \Delta_{11} & \cdots & \Delta_{1\ i_1-1} & 0 & \Delta_{1\ i_1+1} & \cdots & \Delta_{1\ i_2-1} & \Delta_{1\ i_2+1} & \cdots & \Delta_{1r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & & \vdots \\ \hline \Delta_{i'_0\ 1} & \cdots & \Delta_{i'_0\ i_1-1} & 1 & \Delta_{i'_0\ i_1+1} & \cdots & \Delta_{i'_0\ i_2-1} & \Delta_{i'_0\ i_2+1} & \cdots & \Delta_{i'_0\ r} \\ \hline \vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \Delta_{r'1} & \cdots & \Delta_{r'\ i_1-1} & 0 & \Delta_{r'\ i_1+1} & \cdots & \Delta_{r'\ i_2-1} & \Delta_{r'\ i_2+1} & \cdots & \Delta_{r'r} \end{array}\right).$$

Δ° now having Bratteli diagram (79), $\Delta^\circ \in \mathrm{N}(L, L^\circ)$ by Lemma 5.4. Analogous to the first case, one can check that $\ell(\Delta) = 1$, $\Delta^{\circ\prime}\Delta^\circ = \Delta$, and $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$. This proves the lemma. □

5.5. Characterization of direct successors

Theorem 5.6. *Let $\kappa, \kappa' \in \hat{\mathrm{K}}(P)$. The following assertions are equivalent*:

(a) *κ' is a direct successor of κ.*
(b) *There exist representatives L and L' of κ and κ', respectively, such that* $\mathrm{N}(L, L')$ *contains an element of level* 1.
(c) *For any representatives L, L' of κ and κ', respectively*, $\mathrm{N}(L, L')$ *is non-empty and its elements have level* 1.

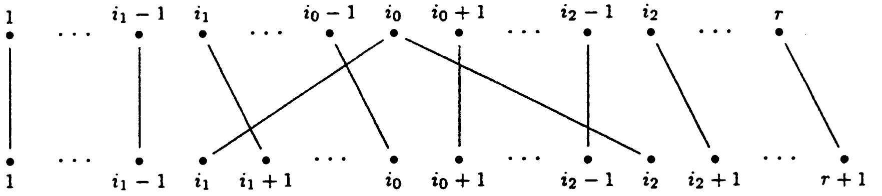
Proof.

(a) ⇒ (c): Let L and L' be representatives of κ and κ', respectively. Since $\kappa \leq \kappa'$ due to Corollary 3.5, there exists $\Delta \in \mathrm{N}(L, L')$. Since $\kappa \neq \kappa'$ due to Lemma 5.3, $\ell(\Delta) \neq 0$. Then Lemma 5.5 implies that there exist $L^\circ \in \mathrm{K}(P)$ and $\Delta^\circ \in \mathrm{N}(L, L^\circ)$, $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$ such that $\Delta = \Delta^{\circ\prime}\Delta^\circ$ and $\ell(\Delta^\circ) = 1$. Let κ° denote the equivalence class of L°. We have $\kappa \leq \kappa^\circ \leq \kappa'$. According to Lemma 5.3, $\kappa \neq \kappa^\circ$. It follows that $\kappa^\circ = \kappa'$. Hence, again due to Lemma 5.3, $\ell(\Delta^{\circ\prime}) = 0$. Then the sharpened version of (63) implies $\ell(\Delta) = \ell(\Delta^\circ) = 1$.

(c) ⇒ (b): Obvious.

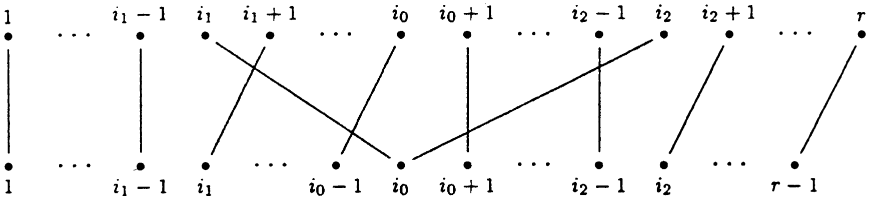
(b) ⇒ (a): Let $L, L' \in \mathrm{K}(P)$ be given as assumed. Let $\kappa^\circ \in \hat{\mathrm{K}}(P)$ such that $\kappa \leq \kappa^\circ \leq \kappa'$. For any representative L° of κ°, there exist $\Delta^\circ \in \mathrm{N}(L, L^\circ)$ and $\Delta^{\circ\prime} \in \mathrm{N}(L^\circ, L')$. Due to Lemma 5.1, $\Delta^{\circ\prime}\Delta^\circ \in \mathrm{N}(L, L')$. Since, by assumption, this set contains an element of level 1, Lemma 5.2 yields $\ell(\Delta^{\circ\prime}\Delta^\circ) = 1$. Then (63) requires that either $\ell(\Delta^\circ) = 0$ or $\ell(\Delta^{\circ\prime}) = 0$. According to Lemma 5.3, in the first case, $\kappa = \kappa^\circ$, whereas in the second case, $\kappa^\circ = \kappa'$. This shows that κ' is a direct successor of κ. □

The Bratteli diagram of an inclusion matrix of level 1: Let $L, L' \in \mathrm{K}(P)$ and let $\Delta \in \mathrm{N}(L, L')$. Assume that $\ell(\Delta) = 1$. Then either there exists i_0 such that $\ell^+_{i_0}(\Delta) = 1$ and $\ell^+_i(\Delta) = 0$ for all $i \neq i_0$ and $\ell^-_{i'}(\Delta) = 0$ for all i', or there exists i'_0 such that $\ell^-_{i'_0}(\Delta) = 1$ and $\ell^-_{i'}(\Delta) = 0$ for all $i' \neq i'_0$ and $\ell^+_i(\Delta) = 0$ for all i. Accordingly, the Bratteli diagram of Δ is given by



(84)

for some $1 \leq i_1 < i_2 \leq r + 1$, or by



(85)

for some $1 \leq i_1 < i_2 \leq r$, respectively. In particular, in the first case, $r' = r + 1$ and in the second case, $r' = r - 1$.

5.6. Generation of direct successors

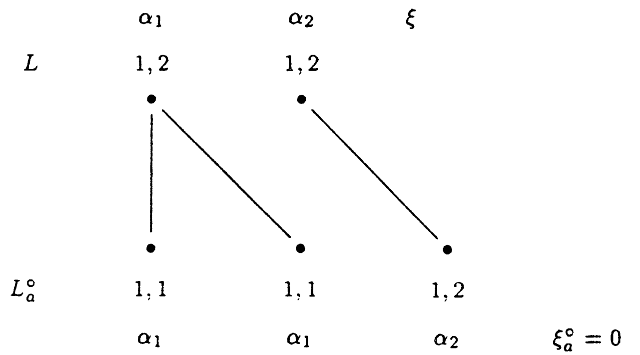
Theorem 5.7. *Let $\kappa \in \hat{\mathrm{K}}(P)$ and let L be a representative of κ. Then the direct successors of κ are obtained by applying all possible splittings and mergings to L and passing to equivalence classes.*

Proof. As an immediate consequence of Lemma 5.4 and Theorem 5.6, any element of $\hat{\mathrm{K}}(P)$ generated in the way proposed is a direct successor of κ. Conversely, let κ' be a direct successor of κ. Choose a representative L' of κ'. Due to Theorem 5.6, $\mathrm{N}(L, L')$ contains an element of level 1. As noted above, the Bratteli diagram of such an element is of the form (84) or (85). By a permutation of the lower vertices we can turn this diagram into (78) or (79), respectively. This corresponds to the passage from L' to another representative L° of κ'. It is immediately seen that the matrix given by the diagram with permuted lower vertices belongs to $\mathrm{N}(L, L^\circ)$. Then Lemma 5.4 implies that L° can be obtained from L by a splitting or a merging, respectively. This proves the theorem. □

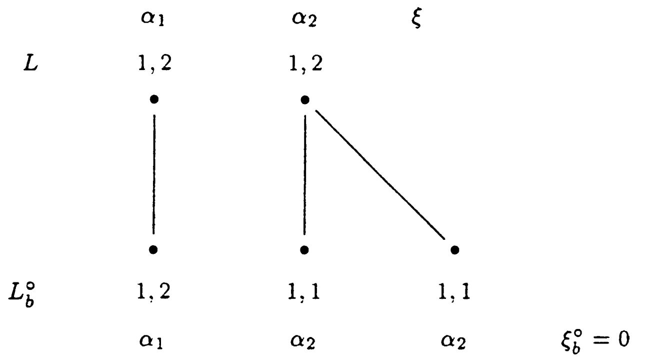
5.7. Example

Let P be a principal SU4-bundle. Let $L \in \mathrm{K}(P)$, $L = (J; \alpha, \xi)$, where $J = (\mathbf{k}, \mathbf{m}) = ((1, 1), (2, 2))$. Then α has components $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2$. (One may wish to recall from the example of Section 3 that SUJ has isomorphism type $\mathbb{Z}_2 \times$ U1.) We are going to determine the direct successors of the equivalence class of L.

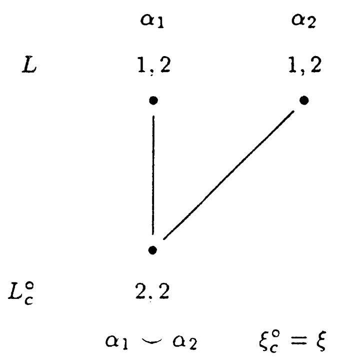
Let us begin with splitting operations. For $i_0 = 1$, the only possible splitting is given by the decomposition $m_1 = 2 = 1 + 1$. It yields $L_a^\circ = (J_a^\circ; \alpha_a^\circ, \xi_a^\circ)$, where $J_a^\circ = ((1, 1, 1), (1, 1, 2))$, $\alpha_a^\circ = (\alpha_1, \alpha_1, \alpha_2)$, and $\xi_a^\circ = 0$. The passage from L to L_a° can be represented conveniently in a Bratteli diagram whose vertices are labeled by the respective quantities k_i, m_i and α_i (rather than by the mere number i):



For $i_0 = 2$, a similar splitting operation creates L_b°, given by the labeled Bratteli diagram



As for merging operations, the only choice for i_1, i_2 is $i_1 = 1,\ i_2 = 2$. This yields L_c°:



Next, we have to pass to equivalence classes. Generically, L_a°, L_b°, and L_c° generate their own classes. However, while L_c° can never be equivalent to L_a° or L_b°, the latter are equivalent iff $\alpha_1 = \alpha_2$. In order to see for which bundles P this can happen, consider Eqs. (10) and (11). The first one requires $\alpha_1^{(2)} = \alpha_2^{(2)}$ to be a torsion element. Then, due to $\alpha_1^{(4)} = \alpha_2^{(4)} = 0$, the second one implies $c_2(P) = 0$. Thus, L_a° and L_b° can be (occasionally) equivalent only if P is trivial.

6. Direct predecessors

In this section, we formulate operations to generate the direct predecessors of any given element of $\hat{\mathrm{K}}(P)$. Direct predecessors are, for our purposes, more interesting than direct successors for at least two reasons. First, they allow one to reconstruct the set $\hat{\mathrm{K}}(P)$ together with its partial ordering from the unique maximal element (which, in terms of Howe subbundles, is given by P itself). Second, on the level of the stratification of the gauge orbit space, predecessors correspond to strata of higher symmetry.

In the preceding section, we have been able to create all direct successors of a given element of $\hat{\mathrm{K}}(P)$ from one and the same representative. This was achieved by using the freedom in the choice of the representatives of the direct successors. Since here we wish to proceed likewise, we have to carry this freedom from the level of successors to that of predecessors. For this reason, the inverted operations are not just splitting and merging read backwards. They rather take the following form. Let $L \in \mathrm{K}(P),\ L = (J; \alpha, \xi)$.

6.1. Inverse splitting

Choose $1 \leq i_1 < i_2 \leq r$ such that $k_{i_1} = k_{i_2}$ and $\alpha_{i_1} = \alpha_{i_2}$. Define sequences of length $(r-1)$

$$\begin{aligned}
\mathbf{k}^\circ &= (k_1, \dots, k_{i_1-1}, k_{i_1}, k_{i_1+1}, \dots, \widehat{k_{i_2}}, \dots, k_r),\\
\mathbf{m}^\circ &= (m_1, \dots, m_{i_1-1}, m_{i_1} + m_{i_2}, m_{i_1+1}, \dots, \widehat{m_{i_2}}, \dots, m_r),\\
\alpha^\circ &= (\alpha_1, \dots, \alpha_{i_1-1}, \alpha_{i_1}, \alpha_{i_1+1}, \dots, \widehat{\alpha_{i_2}}, \dots, \alpha_r).
\end{aligned}$$

We note that g divides the greatest common divisor g° of $\mathbf{m}^\circ$, so that $\varrho_{g^\circ g}$ is well defined. Choose $\xi^\circ \in H^1(M, \mathbb{Z}_{g^\circ})$ such that $\xi = \varrho_{g^\circ g}(\xi^\circ)$ and

$$\beta_{g^\circ}(\xi^\circ) = E^{(2)}_{\tilde{\mathbf{m}}^\circ}(\alpha^\circ). \tag{86}$$

Denote $J^\circ = (\mathbf{k}^\circ, \mathbf{m}^\circ)$ and $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$. We check that $L^\circ \in \mathrm{K}(P)$: by construction, $\mathbf{m}^\circ \cdot \mathbf{k}^\circ = n$ and $\alpha^\circ \in H^{(J^\circ)}(M, \mathbb{Z})$. Due to (86), α° and ξ° obey Eq. (10). Using $\alpha_{i_1} = \alpha_{i_2}$, one can check that $E_{\mathbf{m}^\circ}(\alpha^\circ) = E_{\mathbf{m}}(\alpha)$. Hence, since α obeys Eq. (11), so does α°. This proves $L^\circ \in \mathrm{K}(P)$.

We say that L° arises from L by an inverse splitting of the i_1th and the i_2th member.

6.2. Inverse merging

Choose $1 \leq i_0 \leq r$ such that $k_{i_0} \neq 1$. Choose a decomposition $k_{i_0} = k_{i_0,1} + k_{i_0,2}$ with strictly positive integers $k_{i_0,1}, k_{i_0,2}$. Choose cohomology elements $\alpha_{i_0,1}, \alpha_{i_0,2} \in H_0^{\mathrm{even}}(M, \mathbb{Z})$ such that $\alpha^{(2j)}_{i_0,l} = 0$ for $j > k_{i_0,l},\ l = 1, 2$, and

$$\alpha_{i_0,1} \smile \alpha_{i_0,2} = \alpha_{i_0}. \tag{87}$$

Define sequences of length $(r+1)$

$$\begin{aligned}
\mathbf{k}^\circ &= (k_1, \dots, k_{i_0-1}, k_{i_0,1}, k_{i_0,2}, k_{i_0+1}, \dots, k_r),\\
\mathbf{m}^\circ &= (m_1, \dots, m_{i_0-1}, m_{i_0}, m_{i_0}, m_{i_0+1}, \dots, m_r),\\
\alpha^\circ &= (\alpha_1, \dots, \alpha_{i_0-1}, \alpha_{i_0,1}, \alpha_{i_0,2}, \alpha_{i_0+1}, \dots, \alpha_r),
\end{aligned}$$

as well as

$$\xi^\circ = \xi.$$

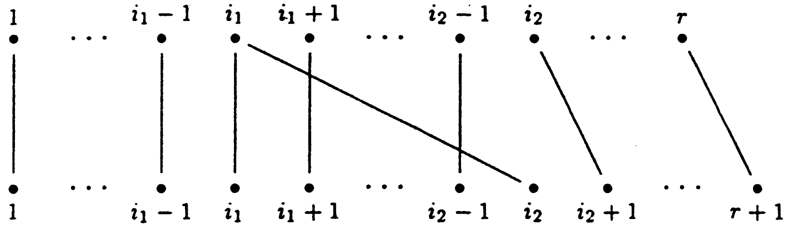
Denote $J^\circ = (\mathbf{k}^\circ, \mathbf{m}^\circ)$ and $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$. To see that $L^\circ \in \mathrm{K}(P)$, we check $\mathbf{m}^\circ \cdot \mathbf{k}^\circ = n$ and $\alpha^\circ \in H^{(J^\circ)}(M, \mathbb{Z})$. Using (87), one can verify that $E_{\mathbf{m}^\circ}(\alpha^\circ) = E_{\mathbf{m}}(\alpha)$.

Consequently, α° obeys Eq. (11). A similar calculation, using, in addition, $g^\circ = g$, shows that $E_{\tilde{\mathbf{m}}^\circ}(\alpha^\circ) = E_{\tilde{\mathbf{m}}}(\alpha)$. Since also $\beta_{g^\circ}(\xi^\circ) = \beta_g(\xi)$, α° and ξ° obey Eq. (10).

We say that L° arises from L by an inverse merging of the i_0th member.

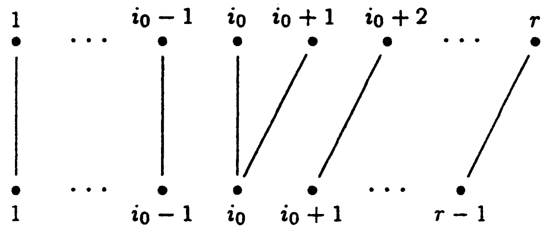
Remark. Like for the operations of splitting and merging, for some of the elements of $\mathrm{K}(P)$, inverse splitting or inverse merging may not be applicable. In particular, it may happen that there does not exist a solution ξ° of Eq. (86).

Lemma 6.1. *Let* $L, L^\circ \in \mathrm{K}(P)$. L° *arises from L by an inverse splitting of the* i_1*th and the* i_2*th member if and only if* $\mathrm{N}(L, L^\circ)$ *contains an element with Bratteli diagram*



(88)

L° *arises from L by an inverse merging of the* i_0*th member if and only if* $\mathrm{N}(L, L^\circ)$ *contains an element with Bratteli diagram*



(89)

Proof. The proof is completely analogous to that of Lemma 5.4 and shall be omitted. □

Theorem 6.2. *Let* $\kappa \in \hat{\mathrm{K}}(P)$ *and let L be a representative of* κ. *Then the direct predecessors of* κ *are obtained by applying all possible inverse splittings and inverse mergings to L and passing to equivalence classes.*

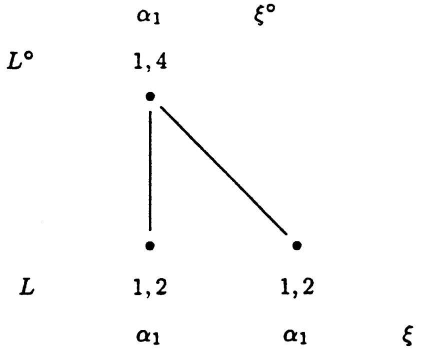
Proof. The proof is completely analogous to that of Theorem 5.7. The only difference is that here we are allowed to pass to another representative of the *predecessor*, i.e., to permute the *upper* vertices in the diagrams (84) and (85), thus arriving at (88) and (89). □

Example. As in Section 5.7, let P be a principal SU4-bundle and let $L \in \mathrm{K}(P)$, $L = (J; \alpha, \xi)$, where $J = ((1, 1), (2, 2))$. We are going to determine the direct predecessors of the equivalence class of L. Inverse splittings can be applied only if $\alpha_1 = \alpha_2$. In this case, for any solution $\xi^\circ \in H^1(M, \mathbb{Z}_4)$ of the system of equations

$$\xi^\circ \bmod 2 = \xi, \tag{90}$$

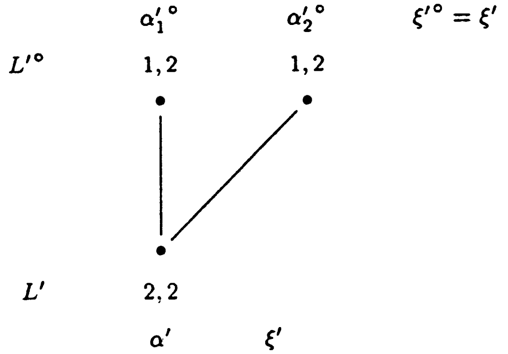
$$\beta_4(\xi^\circ) = \alpha_1^{(2)}, \tag{91}$$

we obtain an element $L^\circ = (J^\circ; \alpha^\circ, \xi^\circ)$, where $J^\circ = ((1), (4))$ and $\alpha^\circ = \alpha_1 = \alpha_2$. The passage from L to L° can be summarized in the labeled Bratteli diagram



that has to be read upwards. Each L° generates its own equivalence class. Due to $k_1 = k_2 = 1$, inverse mergings cannot be applied to L. Thus, in the case $\alpha_1 = \alpha_2$, the direct predecessors of the equivalence class of L are labeled by the solution of Eqs. (90) and (91), whereas in the case $\alpha_1 \neq \alpha_2$ direct predecessors do not exist. Recall from Section 5.7 that the first case can only occur if P is trivial.

As another example, consider an element L' of $\mathrm{K}(P)$, $L' = (J'; \alpha', \xi')$, where $J' = ((2), (2))$. Inverse mergings can be applied and yield elements L'° as follows:



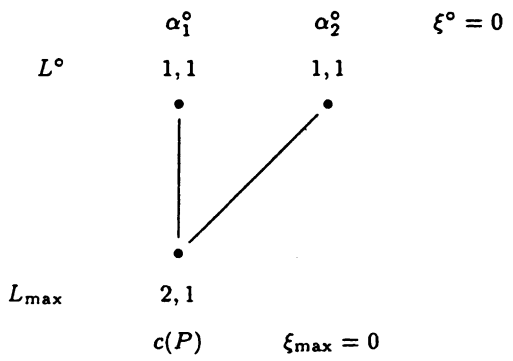
Here $\alpha_i'^\circ = 1 + (\alpha_i'^\circ)^{(2)}$, $i = 1, 2$ such that $\alpha_1'^\circ \smile \alpha_2'^\circ = \alpha'$. When passing to equivalence classes, elements L'° with $\alpha'^\circ = (\alpha_1'^\circ, \alpha_2'^\circ)$ and $\alpha'^\circ = (\alpha_2'^\circ, \alpha_1'^\circ)$ have to be identified. Since L' does not allow inverse splittings, there are no more direct predecessors.

7. Example: gauge orbit types for SU2

The gauge orbit types for SU2, i.e., the set $\hat{\mathrm{K}}(P)$ for a principal SU2-bundle P over M, was calculated in [13] by solving Eqs. (10) and (11) for all J. Here, we are going to recover this result using a different technique that will also yield the partial ordering of orbit types.

A partially ordered set can be reconstructed either: (a) from its minimal elements by successively determining direct successors, or (b) from its maximal elements by successively determining direct predecessors. In the case of $\hat{\mathrm{K}}(P)$, there exists a unique maximal element, namely the class corresponding to the bundle P itself. Minimal elements are, in general, not unique. In fact, their number can be infinite. Thus, the preferred algorithm is (b).

The unique representative of the maximal element of $\mathrm{K}(P)$ is $L_{\max} = (J_{\max}; \alpha_{\max}, \xi_{\max})$, where $J_{\max} = ((2), (1))$, $\alpha_{\max} = c(P)$, and $\xi_{\max} = 0$. Inverse mergings yield elements L°:



where $\alpha_i^\circ = 1 + (\alpha_i^\circ)^{(2)}$ such that $\alpha_1^\circ \smile \alpha_2^\circ = c(P)$. Sorting by degree yields the equations $(\alpha_1^\circ)^{(2)} + (\alpha_2^\circ)^{(2)} = 0$ and $(\alpha_1^\circ)^{(2)} \smile (\alpha_2^\circ)^{(2)} = c_2(P)$. We parameterize

$$(\alpha_1^\circ)^{(2)} = \alpha^{(2)}, \qquad (\alpha_2^\circ)^{(2)} = -\alpha^{(2)},$$

where $\alpha^{(2)} \in H^2(M, \mathbb{Z})$ has to obey

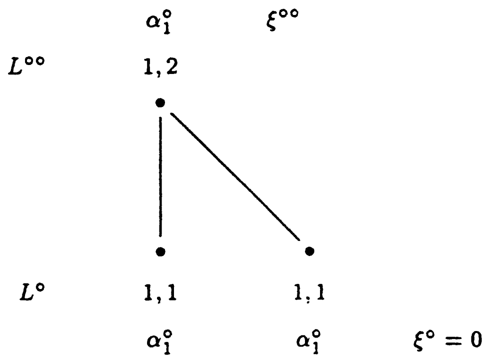
$$-\alpha^{(2)} \smile \alpha^{(2)} = c_2(P). \tag{92}$$

The passage to equivalence classes leads to an identification of solutions $\alpha^{(2)}$ and $-\alpha^{(2)}$. We note that the Howe subgroup labeled by $J = ((1, 1), (1, 1))$ is the toral subgroup U1 of SU2 and that the parameter $\alpha^{(2)}$ is just the first Chern class of the corresponding reduction of P. By virtue of this transliteration, Eq. (92) coincides with the result given in [9].

Next, consider the direct predecessors of the classes generated by L°. Inverse mergings cannot be applied. Inverse splittings can be applied provided $\alpha_1^\circ = \alpha_2^\circ$, i.e., $\alpha^{(2)} = -\alpha^{(2)}$. Then for any solution $\xi^{\circ\circ} \in H^1(M, \mathbb{Z}_2)$ of the equation

$$\beta_2(\xi^{\circ\circ}) = \alpha^{(2)}, \tag{93}$$

we obtain an element $L^{\circ\circ}$:



Each of these elements generates its own equivalence class.

Note that the Howe subgroup labeled by $J = ((1), (2))$ is the center $\mathbb{Z}_2$ of SU2 and that $\xi^{\circ\circ}$ is the natural characteristic class for principal $\mathbb{Z}_2$-bundles over M, see [15, Section 13].

Now let us draw Hasse diagrams of $\hat{\mathrm{K}}(P)$ for specific space–time manifolds M. In the following, vertices stand for the elements of $\hat{\mathrm{K}}(P)$ and edges indicate the relation 'left vertex $\leq$ right vertex'. When viewing the elements of $\hat{\mathrm{K}}(P)$ as Howe subbundles, the vertex on the r.h.s. represents the class corresponding to P itself, the vertices in the middle and on the l.h.s. represent reductions of P to the Howe subgroups U1 and $\mathbb{Z}_2$, respectively. When viewing the elements of $\hat{\mathrm{K}}(P)$ as orbit types, or strata of the gauge orbit space, the vertex on the r.h.s. represents the generic stratum, whereas the vertices in the middle and on the l.h.s. represent U1-strata and SU2-strata. Here the names U1-stratum and SU2-stratum mean that the stratum consists of (orbits of) connections whose stabilizers are isomorphic to U1 or SU2, respectively.

$M = \mathrm{S}^4$: Since $H^2(M, \mathbb{Z}) = 0$, Eq. (92) can be solved iff $c_2(P) = 0$, i.e., iff P is trivial. The solution is $\alpha^{(2)} = 0$. Then Eq. (93) is trivially satisfied by $\xi^{\circ\circ} = 0$. Due to $H^1(M, \mathbb{Z}_2) = 0$, there are no more solutions. Thus, in the case $c_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is

If $c_2(P) \neq 0$, on the other hand, $\hat{\mathrm{K}}(P)$ is trivial, meaning that it consists only of the class corresponding to P itself.

On the level of gauge orbit types, the result means that in the sector of vanishing topological charge the gauge orbit space decomposes into the generic stratum, an U1-stratum, and an SU2-stratum. If, on the other hand, a topological charge is present, only the generic stratum survives.

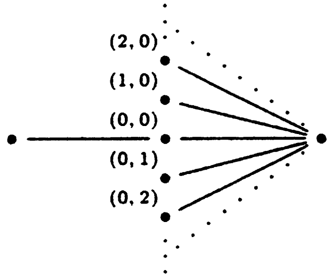
$M = \mathrm{S}^2 \times \mathrm{S}^2$: To perform the first step in the reconstruction procedure, let 1_{S^2} and $\gamma_{\mathrm{S}^2}^{(2)}$ be generators of $H^0(\mathrm{S}^2, \mathbb{Z})$ and $H^2(\mathrm{S}^2, \mathbb{Z})$, respectively. Due to the Künneth theorem, $H^2(M, \mathbb{Z})$ is generated by $\gamma_{\mathrm{S}^2}^{(2)} \times 1_{\mathrm{S}^2}$ and $1_{\mathrm{S}^2} \times \gamma_{\mathrm{S}^2}^{(2)}$, whereas $H^4(M, \mathbb{Z})$ is generated by $\gamma_{\mathrm{S}^2}^{(2)} \times \gamma_{\mathrm{S}^2}^{(2)}$. Here $\times$ denotes the cohomology cross-product. Writing

$$\alpha^{(2)} = a\gamma_{\mathrm{S}^2}^{(2)} \times 1_{\mathrm{S}^2} + b 1_{\mathrm{S}^2} \times \gamma_{\mathrm{S}^2}^{(2)} \tag{94}$$

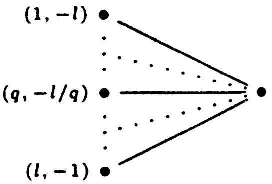
with $a, b \in \mathbb{Z}$, Eq. (92) becomes

$$-2ab\gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2} = c_2(P). \tag{95}$$

If $c_2(P) = 0$, there are two series of solutions: $a = 0$ and $b \in \mathbb{Z}$ as well as $a \in \mathbb{Z}$ and $b = 0$. Due to $H^1(M, \mathbb{Z}_2) = 0$, Eq. (93) tells us that out of the elements just obtained only that labeled by $a = b = 0$ has a direct predecessor. Thus, in the case $c_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



The vertices in the middle are labeled by the corresponding values of (a, b). Note that passage to equivalence classes requires identification of solutions (a, b) and $(-a, -b)$. If $c_2(P) = 2l\gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2}$, $l \neq 0$, then the solutions of (95) are $a = q$ and $b = -l/q$, where q runs through the (positive and negative) divisors of l. For none of these solutions, (93) is solvable. Hence, here the Hasse diagram is



where due to the identification $(a, b) \sim (-a, -b)$, q runs through the positive divisors of l only. If $c_2(P) = (2l + 1)\gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2}$, (95) has no solutions, so that $\hat{\mathrm{K}}(P)$ is trivial.

Finally, the interpretation of the result in terms of strata of the gauge orbit space is similar to that for space–time manifold $M = \mathrm{S}^4$ above.

$M = \mathrm{L}^3_{2p} \times \mathrm{S}^1$: Recall that $H^1(\mathrm{L}^3_{2p}, \mathbb{Z}) = 0$ and $H^2(\mathrm{L}^3_{2p}, \mathbb{Z}) \cong \mathbb{Z}_{2p}$. Let $\gamma^{(2)}_{\mathrm{L},\mathbb{Z}}$ be a generator of $H^2(\mathrm{L}^3_{2p}, \mathbb{Z})$ and let $1_{\mathrm{S}^1,\mathbb{Z}}$ be the generator of $H^0(\mathrm{S}^1, \mathbb{Z})$. Due to the Künneth theorem, $H^2(M, \mathbb{Z})$ is generated by $\gamma^{(2)}_{\mathrm{L},\mathbb{Z}} \times 1_{\mathrm{S}^1,\mathbb{Z}}$. We write

$$\alpha^{(2)} = a\gamma^{(2)}_{\mathrm{L},\mathbb{Z}} \times 1_{\mathrm{S}^1,\mathbb{Z}}. \tag{96}$$

Due to $2p\gamma^{(2)}_{\mathrm{L},\mathbb{Z}} = 0$, $\alpha^{(2)} \smile \alpha^{(2)} = 0$. Hence, Eq. (92) is solvable iff $c_2(P) = 0$, in which case the solutions are given by $a \in \mathbb{Z}_{2p}$. Since when passing to equivalence classes, we have to identify $\alpha^{(2)}$ and $-\alpha^{(2)}$, i.e., a and $-a$, the direct predecessors are labeled by elements of $\mathbb{Z}_p$.

Next, consider the second step of the reconstruction procedure. Let $1_{\mathrm{L},\mathbb{Z}_2}$, $\gamma^{(1)}_{\mathrm{L},\mathbb{Z}_2}$, and $\gamma^{(1)}_{\mathrm{S}^1,\mathbb{Z}}$ be generators of $H^0(\mathrm{L}^3_{2p},\mathbb{Z}_2)$, $H^1(\mathrm{L}^3_{2p},\mathbb{Z}_2)$, and $H^1(\mathrm{S}^1,\mathbb{Z})$, respectively. Then, again due to the Künneth theorem, $H^1(M,\mathbb{Z}_2)$ is generated by $\gamma^{(1)}_{\mathrm{L},\mathbb{Z}_2}\times 1_{\mathrm{S}^1,\mathbb{Z}}$ and $1_{\mathrm{L},\mathbb{Z}_2}\times\gamma^{(1)}_{\mathrm{S}^1,\mathbb{Z}}$. Moreover, one can check that

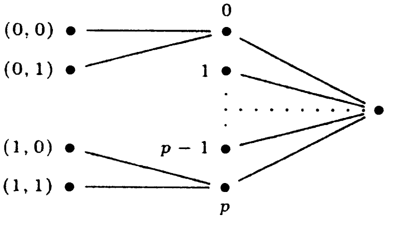
$$\beta_2(\gamma^{(1)}_{\mathrm{L},\mathbb{Z}_2}\times 1_{\mathrm{S}^1,\mathbb{Z}}) = p\gamma^{(2)}_{\mathrm{L},\mathbb{Z}},\qquad \beta_2(1_{\mathrm{L},\mathbb{Z}_2}\times\gamma^{(1)}_{\mathrm{S}^1,\mathbb{Z}}) = 0. \tag{97}$$

Decomposing $\xi^{\circ\circ} = a_{\mathrm{L}}\gamma^{(1)}_{\mathrm{L},\mathbb{Z}_2}\times 1_{\mathrm{S}^1,\mathbb{Z}} + a_{\mathrm{S}}1_{\mathrm{L},\mathbb{Z}_2}\times\gamma^{(1)}_{\mathrm{S}^1,\mathbb{Z}}$ and using (96) and (97), (93) becomes

$$pa_{\mathrm{L}} = a.$$

Thus, only the elements labeled by $a = 0$ and $a = p$ have direct predecessors. These are given by the values $a_{\mathrm{L}} = 0$, $a_{\mathrm{S}} = 0, 1$ and $a_{\mathrm{L}} = 1$, $a_{\mathrm{S}} = 0, 1$, respectively.

As a result, in the case $c_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



Here the vertices on the l.h.s. are labeled by $(a_{\mathrm{L}}, a_{\mathrm{S}})$, whereas those in the middle are labeled by a. In the case $c_2(P)\neq 0$, $\hat{\mathrm{K}}(P)$ is trivial. Again, the interpretation in terms of strata of the gauge orbit space goes along the lines of the case $M = \mathrm{S}^4$ above.

To conclude, let us remark that, while for SU2 the picture is relatively simple, already for SU3 the partial ordering becomes rather involved, and the Hasse diagrams representing it are very complex.

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